Solutions - Winter 2020 AMATYC Student Mathematics League Contest

1. Let AC denote the volume (in ml.) poured from A to C. Similarly for BC, AD, and BD.
Then the salt concentrations give:

$$.20\,AC + .50\,BC = .30(100) \quad , \quad .20\,AD + .50\,BD = .45(200)$$

that is:

$$2AC + 5BC = 300 \quad , \quad 2AD + 5BD = 900$$

and the volumes satisfy:

$$AC + BC = 100 \quad , \quad AD + BD = 200.$$

solve this system   solve this system

$$2(100 - BC) + 5BC = 300 \qquad 2(200 - BD) + 5BD = 900$$
$$3BC = 100 \qquad 3BD = 500$$
$$BC = 100/3 \qquad BD = 500/3.$$

Thus, the volume remaining in B is $800 - 100/3 - 500/3 = 600$ ml.

2. Let the first and second rectangles measure p by q and r by s, respectively, with $p<q$ and $r<s$. Then either $p<q<r<s$ or $p<r<q<s$.

If $p<q<r<s$ then $(p, q, r, s) = (p, p+2, p+4, p+6)$
so $2p + 2(p+2) + 44 = 2[2(p+4) + 2(p+6)]$
$4p + 48 = 8p + 40$
$p = 2$, contradicting that p is odd.

If $p<r<q<s$ then $(p, r, q, s) = (p, p+2, p+4, p+6)$
so $2p + 2(p+4) + 44 = 2[2(p+2) + 2(p+6)]$
$4p + 52 = 8p + 32$
$p = 5$.

Thus, the sum of the areas is:

$$p(p+4) + (p+2)(p+6) = 5(9) + 7(11) = 122 < 150.$$

3. $a = \sqrt{2020 - b^2 - c^4}$, where $c^4 < 2020$ so $c < 2020^{1/4} \doteq 6.7$

On a TI-84 calculator: $\backslash Y1 = \sqrt{(2020 - X^2 - C\wedge 4)}$
$1 \to C$

Using TABLE, there are no integers X and $Y1$ with $X < Y1$.
Repeat this for $2 \to C$, $3 \to C$, ..., $6 \to C$.

continued

(3, cont'd)

At 6 → C, observe X = 18, Y1 = 20.

Thus, $a + b + c = 20 + 18 + 6 = 44$.

Alternatively, a calculator program can be written to search for the solution.

4. The two examples given could be written "$x = 3$" and "$x + 2 = 5$". Let the term "addition" refer to adding a weight to the side opposite the object, and "subtraction" refer to adding a weight to the side with the object.

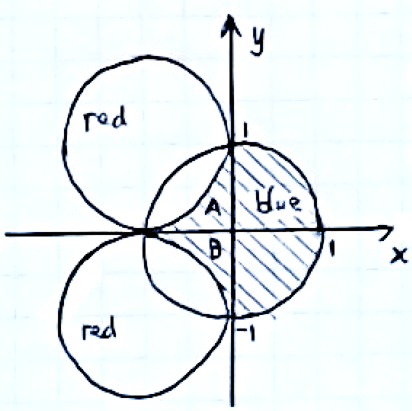
Clearly we could use binary weights 1, 2, 4, 8, ..., 32 because every positive integer has a (unique) binary representation. But this is less than optimal because it uses only addition, not taking advantage of subtraction.

Every positive integer has a (unique) base-3 representation, and since $2_3 = 10_3 - 1$, the units digit 2 in that representation can be replaced with

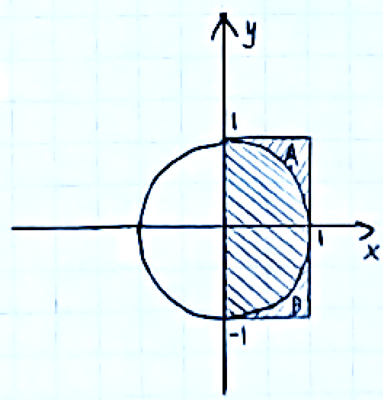
adding $3 = 10_3$ and subtracting 1. Similarly for a digit 2 in any other place value. Thus, a single copy of each power of 3 suffices to represent every positive integer using addition and subtraction. For objects up to 40 pounds the set of weights needed is 1, 3, 9, 27 (four weights).

[This is optimal. For example, in base 4 we have $2_4 = 10_4 - 2_4$, which requires two copies of the weight 1.]

5.



By dissection (cutting and pasting pieces A and B), the remaining blue region has the same area as a 1×2 rectangle, namely 2.



6. Do an experiment: Let $K = 1 + 2 + 3 + \cdots + 8 = \frac{8 \cdot 9}{2} = 6^2$, so $D = 8$ and $\sqrt{K} = 6$.

$[1 + 2 + \cdots + 5] - [7 + 8] = 15 - 15 = 0$. This rules out choices

A, B, D, E and so C is correct.

continued

6. (continued) Another way –
Recall that $1+2+\cdots+n = \dfrac{n(n+1)}{2}$; e.g., $K = 1+2+\cdots+D = \dfrac{D(D+1)}{2}$.

So $\left[1+2+\cdots+(\sqrt{K}-1)\right] - \left[(\sqrt{K}+1)+(\sqrt{K}+2)+\cdots+D\right]$

$= 2\left[1+\cdots+(\sqrt{K}-1)\right] - 1\left[1+\cdots+(\sqrt{K}-1)\right] + (\sqrt{K}-\sqrt{K}) - \left[(\sqrt{K}+1)+\cdots+D\right]$

$= 2\left[1+\cdots+(\sqrt{K}-1)\right] + \sqrt{K} - \left\{\left[1+\cdots+(\sqrt{K}-1)\right] + \sqrt{K} + \left[(\sqrt{K}+1)+\cdots+D\right]\right\}$

$= 2\dfrac{(\sqrt{K}-1)\sqrt{K}}{2} + \sqrt{K} - \dfrac{D(D+1)}{2}$

$= (\sqrt{K}-1)\sqrt{K} + \sqrt{K} - K$

$= K - \sqrt{K} + \sqrt{K} - K = 0.$

7. Let t denote elapsed time in mins. The tortoise loses its 900-m. lead on Achilles after going a distance $1000t = 10t + 900$, whose solution is $t = 10/11$. Achilles loses his 100-m lead on the hare after going a distance $1000t + 100 = 1050t$, whose solution is $t = 2$. Thus, Achilles holds the lead for $2 - 10/11 = 12/11$ min.

8. Eqn. A: $D + N + Q = 62$ coins
Eqn. B: $10D + 5N + 25Q = 830$ cents.

To eliminate N, take $\frac{1}{5}B - A$: $D + 4Q = 104$, so $Q \leq 104/4 = 26$.
To eliminate D, take $2A - \frac{1}{5}B$: $N - 3Q = -42$, so $Q \geq 42/3 = 14$.

Each value of Q from 14 to 26 gives one solution, so the sum of the possible values of N is the sum of $3Q-42$ for $14 \leq Q \leq 26$, i.e.:
$$\sum_{Q=14}^{26}(3Q-42) = 3\sum_{Q=14}^{26}(Q-14) = 3\sum_{Q=0}^{12} Q = 3\,\frac{12(13)}{2} = 234.$$

9. Let H and M denote the hour and minute hands, respectively.

Suppose that H leads M by 30° at the first observation. Then exactly 10 mins. later, M is in the same position but H has advanced $\frac{10 \text{ mins.}}{720 \text{ mins.}} \times 360^\circ = 5^\circ$, so the angle cannot have changed from 30° to 85°. (Note that the speed ratio of the two hands is $5^\circ/60^\circ = 1/12$.)
Thus, M leads H by 30° at the first observation. Let m and h denote the angles of M and H, respectively, measured in degrees (clockwise positively) from 12:00. Then $m/12$ is the number of degrees by which H has passed 3:00, i.e., passed 90°. Thus, $m = 30 + \underbrace{\left(90 + m/12\right)}_{h}$, whose root is $m = \frac{1440}{11}^\circ$.

continued

9. (cont'd) $\dfrac{1440}{11}^\circ \times \dfrac{60\text{ mins}}{360^\circ} = 21.\overline{81}\text{ mins.} \doteq 21\text{ mins } 49\text{ secs},$

so the answer is 3:21:49.

10. $\left.\begin{array}{l} A\#B = AB + BA \\ B\#A = BA + AB \end{array}\right\}$ so # is commutative

$$A\#(B\#C) = A\#(BC+CB) = A(BC+CB) + (BC+CB)A = ABC + \overbrace{ACB + BCA} + CBA$$
$$(A\#B)\#C = (AB+BA)\#C = (AB+BA)C + C(AB+BA) = ABC + \underbrace{BAC + CAB} + CBA$$

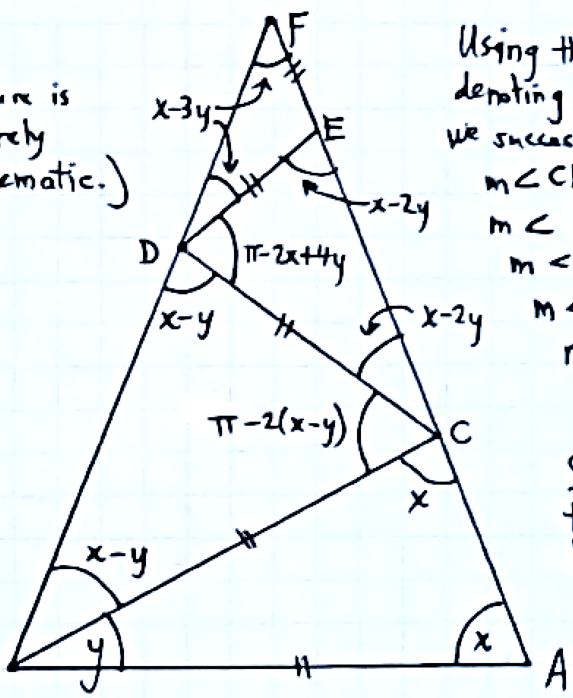
so # is not associative.

11. Let T, F, M denote knight, knave, and spy, respectively ('true', 'false', 'maybe').

person	statement	A	B	C	D	E
X	Z = M	M	M	F	T	F
Y	Z = M	T	F	T	M	M
Z	Z = M	F	T	M	F	T

All three people say Z is a spy, and one of them is a knight telling the truth, so any choice (A, B, D, E) without Z as spy can be ruled out (the circled item in each column is inconsistent with the person and statement on that row). This leaves C as correct.

12. (Figure is merely schematic.)



Using the isosceles triangles, if we start by denoting $m\angle FAB = x$ and $m\angle ABC = y$, in radians, then we successively find:

$m\angle CBD = m\angle CDB = x-y$

$m\angle BCD = \pi - 2(x-y)$

$m\angle DCE = \pi - x - [\pi - 2(x-y)] = x - 2y$

$m\angle CDE = \pi - 2(x-2y) = \pi - 2x + 4y$

$m\angle EDF = m\angle EFD = \pi - (x-y) - (\pi - 2x + 4y)$
$= m\angle AFB = x - 3y.$

Since ΔABC and ΔAFB are similar (isosceles triangles with base angles of measure x), we have $x - 3y = y$, so $y = \frac{1}{4}x$.

In these two triangles, the angles' total measure is $x + x + \frac{1}{4}x = \pi$, so $x = \frac{4}{9}\pi$, and then $y = \frac{1}{4}\left(\frac{4}{9}\pi\right) = \pi/9$ radian or 20°.

13. Based on experiment, we find that the first two 5-digit palindromes that are multiples of 37 are 10101 and 11211. This suggests that we can add 1110 to any such number to produce another, provided that there are no "carries".

From 10101, we can add 1110 eight times before a carry, so this gives 9 solutions.
" 20202 " " " " seven " " " " " " " " 8 "
⋮
" 90909 " " " " zero " " " " " " " " 1 "

Thus, there are $9 + 8 + 7 + \cdots + 1 = \frac{9(10)}{2} = 45$ solutions.

14. · removable discontinuity at $x = 2$ ⟶ factor $x-2$ in numerator and denominator
· vertical asymptote $x = 7$ ⟶ factor $x-7$ in denominator
· roots $x = 4$ and $x = -3$ ⟶ factors $x-4$ and $x+3$ in numerator

Thus $g(x) = \dfrac{(x-2)(x-4)(x+3)}{(x-2)(x-7)}$, so $N + M = (2\cdot4\cdot3) + (2\cdot7) = 38$.

[Note that except at $x = 2$, $g(x) = \dfrac{(x-4)(x+3)}{x-7} = \dfrac{x^2-x-12}{x-7} = x + 6 + \dfrac{30}{x-7}$, so $y = x+6$ is a slant asymptote.]

15. Based on experiments such as $5^3 = 125$ and $5^4 = 625$, we observe that 5^n ends in 125 or 625 if n is odd or even, respectively, so 5^{74} ends in 625. Since $625 \times 163 = 101{,}875$ the answer is 8.

16. Since 18 is even, the factors must pair off into 9 pairs, nested together with $1 \times M$ on the outside, and the innermost pair surrounding $\sqrt{M}$.

Since 9 is close to $\sqrt{200} \doteq 14$, the factors must be rather closely packed near 1. Guided by this, we are led to these 18 factors:

1 2 3 4 5 6 9 10 12 15 18 20 30 36 45 60 90 180

pairs are shown with brackets

$1 + 2 + 3 + 4 + \cdots + 90 + 180 = 546$.

17. $\begin{cases} rs + t = 14 \\ r + st = 13 \end{cases}$ The system is linear in the variables r and t. So, solve for r and t by any preferred method. One example:

Multiply the first equation by s and subtract the second equation, then solve for:

$$r = \frac{14s-13}{s^2-1}.$$

Multiply the second equation by s and subtract from the first equation, the solve for:

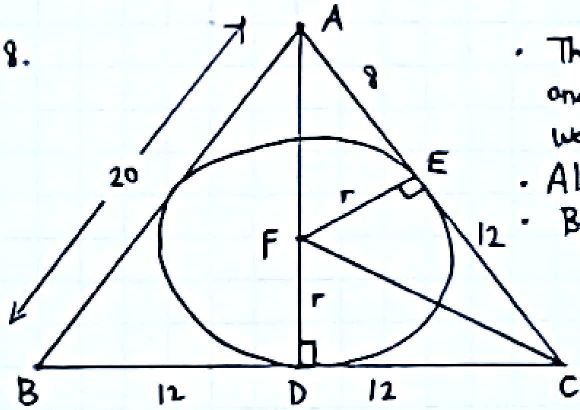
$$t = \frac{13s-14}{s^2-1}.$$

Then $r + t = \frac{27(s-1)}{s^2-1} = \frac{27}{s+1}$. If r and t are integers, then $\frac{27}{s+1}$ is an integer, so $s = 0, 2, 8$ or 26. Now using the earlier equations for r and t:

$s = 0 \longrightarrow r = 13, t = 14$
$s = 2 \longrightarrow r = 5, t = 4$
$s = 8 \longrightarrow r = 11/7$, not an integer
$s = 26 \longrightarrow r = 13/25$, not an integer.

Thus, there are two triples (r, s, t), viz., $(13, 0, 14)$ and $(5, 2, 4)$.

18.



- The base bisects into segments of length 12, and by congruence of $\triangle CDF$ and $\triangle CEF$, we have $EC = 12$, so $AE = 20 - 12 = 8$.
- Also, $AD = \sqrt{20^2 - 12^2} = 16$.
- By similarity of $\triangle ADC$ and $\triangle AEF$, we have:

$$\frac{16}{12} = \frac{8}{r}, \text{ so } r = 6.$$

19. For $|x| \leq 2$ the equation becomes:

$$(2\sin x)\cos(x\sin x) + 2\sin x = 2\sin x$$
$$(2\sin x)\cos(x\sin x) = 0$$
$$\sin x = 0 \quad \text{or} \quad x\sin x = \pm\pi/2$$
$$x = 0 \quad \text{or} \quad x = \pm\pi/2 \text{ only, if } |x| \leq 2.$$

For $|x| > 2$ the equation becomes:

$$2\cos x + 2\sin x = 2$$
$$\cos x + \sin x = 1$$
$$(\cos x)(\sin \pi/4) + (\sin x)(\cos \pi/4) = \sqrt{2}/2$$
$$\cos(x + \pi/4) = \sqrt{2}/2$$
$$x = \tfrac{-3}{2}\pi \text{ only, if } 2 < |x| < 2\pi.$$

Thus there are four solutions: $0, \pm\pi/2, \frac{-3}{2}\pi$.

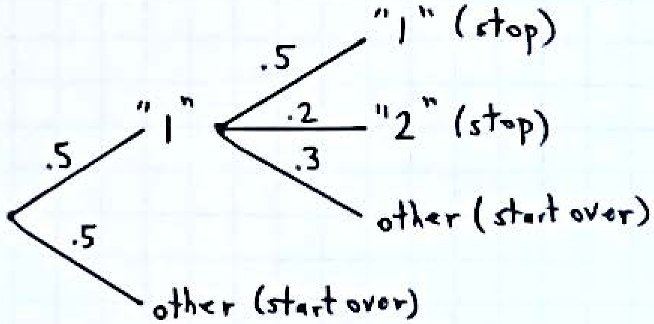
continued

19. (cont'd) Another way: On TI-84 calculator, enter:

$$\backslash Y1 = 2\sin(X) * (\text{abs}(X) \le 2) + 2 * (\text{abs}(X) > 2)$$
$$\backslash Y2 = Y1 * \cos(X * Y1/2) + 2\sin(X) - Y1$$

Graph Y2 in the window $(-2\pi, 2\pi)$ and count the zeros : four.

20.



no. of rolls	probability
2	$(.5)(.5)+(.5)(.2)$
$2+E$	$(.5)(.3)$
$1+E$	$.5$

The expected value of a variable is the weighted average of its possible values, using their probabilities as weights:

$$E = E(x) = \sum x \Pr(x) .$$

The probability tree diagram above shows that we can calculate this E recursively : each "start over" prolongs the roll sequence by 1 or 2 rolls.

$$E = 2\left[(.5)(.5)+(.5)(.2)\right] + (2+E)(.5)(.3) + (1+E)(.5)$$

$$E = .7 + (.3 + .15E) + (.5 + .5E)$$

$$.35E = 1.5$$
$$E = 30/7 .$$