Solutions - October/November 2015 AMATYC Student Mathematics League Contest

1. $\left(\frac{\$6}{10 \text{ apples}} - \frac{\$10}{20 \text{ apples}}\right)(25 \text{ apples}) = \$2.50$

2. Slopes $\frac{-a}{12} \cdot \frac{a}{3} = -1$

$$a^2 = 36$$
$$a = 6$$

3. Rates are additive: $\frac{1 \text{ fence}}{b \text{ hours}} + \frac{1 \text{ fence}}{c \text{ hours}} + \frac{1 \text{ fence}}{d \text{ hours}} = \frac{1 \text{ fence}}{1 \text{ hour}}$

$$1/b + 1/c + 1/d = 1$$

The only distinct-positive-integers solution is $1/2 + 1/3 + 1/6 = 1$, where $b + c + d = 2 + 3 + 6 = 11$.

4. We need a number of students $n = 6x - 2$ that is divisible by 4 and 5.
To find an example, use TABLE with $Y1 = (6X-2)/20$.
The first example is $X = 7$, making $n = 40$ and $n+4 = 44$, divisible by 11.
The next example is $X = 17$, making $n = 100$, too large for the class.

5. $x = 12 \rightarrow 24 + p = q \rightarrow p - q = -24$

$3x + q = p \rightarrow 3x = p - q \rightarrow 3x = -24 \rightarrow x = -8.$

6. The number of coins q, d, n, p satisfies:

$$\begin{cases} q + d + n + p = 42 \\ 25q + 10d + 5n + p = 100 \\ q, d, n, p \geq 1 \end{cases}$$

$>$ subtract, $24q + 9d + 4n = 58$

Since $24(2) + 9(1) + 4(1) > 58$, $q = 1$.

Thus, $\begin{cases} d + n + p = 41 \\ 10d + 5n + p = 75 \end{cases}$ $>$ subtract, $9d + 4n = 34$ so $d \leq 3$.

If $d = 1$ then $4n = 25$, not an integer n.
If $d = 3$ then $4n = 7$, not an integer n.
Thus $d = 2$ so $4n = 16$, so $n = 4$ and $d + n = 6$.

7. $\left.\begin{array}{l} x=-1 \longrightarrow 2 = -a+b-c+d \\ x=1 \longrightarrow 2 = a+b+c+d \\ x=2 \longrightarrow 2 = 8a+4b+2c+d \\ x=3 \longrightarrow 10 = 27a+9b+3c+d \end{array}\right\}$ Solve by any method, such as Gauss-Jordan or rref $\longrightarrow$ $\begin{array}{l} a = 1 \\ b = -2 \\ c = -1 \\ d = 4 \end{array}$

$$P(4) = 1(4^3) - 2(4^2) - 1(4) + 4(1) = 32.$$

Another way: STAT EDIT $(-1,2), (1,2), (2,2), (3,10)$
STAT CALC CubicReg gives $a=1, b=-2, c=-1, d=4$. Etc.

Another way: MATRIX $\begin{bmatrix} 64 & 16 & 4 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 \\ 8 & 4 & 2 & 1 \\ 27 & 9 & 3 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 2 \\ 2 \\ 10 \end{bmatrix}$ gives 32.

8. $\begin{cases} a^2+b^2+c^2 = 17^2 \longrightarrow c = 8 \\ a^2+b^2 = 15^2 \\ a = 9 \longrightarrow b = 12 \end{cases}$ $\therefore V = abc = 9 \cdot 12 \cdot 8 = 864.$

9. $\begin{cases} x+y+z = 25 \longrightarrow x \geq 13 \\ x > 2y \\ x < 3z \\ y > z \end{cases}$ $x + y > 2y + z \longrightarrow x > y + z$

Checking shows that no $(13, y, z)$ satisfies all constraints.

Next, $(14, 6, 5)$ works, so $z = \$5$.

10. Pick the simplest example: $(1,1), (-1,1)$, intercept $1 = ab$.

Another way: $(a, a^2), (-b, b^2)$ Slope $\frac{a^2-b^2}{a+b} = a-b$,

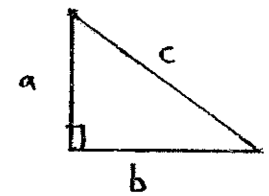
Line $y - a^2 = (a-b)(x-a)$

$$x = 0 \longrightarrow y - a^2 = (a-b)(-a)$$
$$y = a^2 + (-a^2+ab) = ab.$$

Another way: Double False Position $\begin{array}{ccc} a & & a^2 \\ | & \times & \\ -b & & b^2 \end{array}$ $\frac{b^2a-(-ba^2)}{a-(-b)} = \frac{ab(b+a)}{a+b}$
$= ab.$

11. $a + b + c = \frac{1}{2}ab$

$2a + 2b + 2c = ab$



$a^2 + b^2 = c^2$

$2c = ab - 2a - 2b$

$\cancel{4c^2} = a^2b^2 - 4a^2b - 4ab^2 + \cancel{4a^2} + \cancel{4b^2} + 8ab$

$0 = ab(ab - 4a - 4b + 8)$

$0 = ab - 4a - 4b + 8$

$4b - 8 = ab - 4a$

$4b - 8 = a(b-4)$

$a = \dfrac{4b-8}{b-4}$ \Y1 = (4X−8)/(X−4)

TABLE to observe the only positive integer solutions: $(5,12), (12,5) \longrightarrow \frac{1}{2}ab = 30$ or $(6,8), (8,6) \longrightarrow \frac{1}{2}ab = 24$,

so the largest area / perimeter is 30.

12. \Y1 = √(4X+9) + √(9X+1)

TABLE to observe the only nonnegative integer solutions at X = 0, 40.

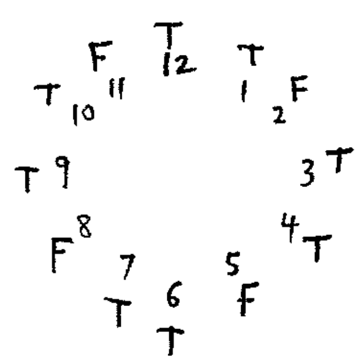
$n = 0 \rightarrow 9 = a^2 \rightarrow a = 3$, $1 = b^2 \rightarrow b = 1$

$n = 40 \rightarrow 169 = a^2 \rightarrow a = 13$, $361 = b^2 \rightarrow b = 19$

Answer, $3 + 1 + 13 + 19 = 36$.

13. Take a simple case like N = 1:

Let T = knight,
F = knave.



Starting with T at 1, we know exactly one neighbor (say 2) is F.

That means 0 or 2 neighbors of 2 are T, so 3 is T.

Continuing like this, we find that the triplet T F T repeats any number of times.

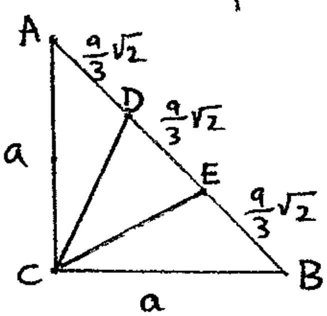
For each triplet there is 1 F, thus for each 12-let there are 4 F's, thus for 12N people there are 4N knaves.

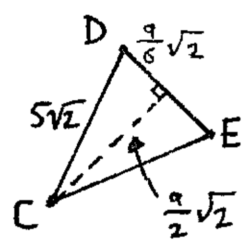
14. If a digit is known to be in a certain position, then the other 5 digits could be anywhere, giving $5! = 120$ possibilities (including repeats due to the two 1's), or $120/2! = 60$ distinct possibilities.

Depending on the position, a digit "d" contributes to the grand total either d or $10d$ or $100d$, etc.

The grand total is thus $60(1+10+100+1000+10000+100000)(2+5+8+1+3+1)$
$= 60(111,111)(20)$
$= 133,333,200$.

15. Use the simplest case, isosceles:



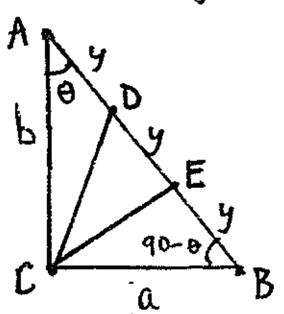


Pythagorean Theorem,

$$CD^2 = \left(\frac{a}{6}\sqrt{2}\right)^2 + \left(\frac{a}{2}\sqrt{2}\right)^2$$
$$(5\sqrt{2})^2 = a^2/18 + a^2/2$$
$$50 = \frac{5}{9}a^2$$
$$a^2 = 90$$
$$a = 3\sqrt{10}$$
$$AB = 3\sqrt{10}\cdot\sqrt{2} = 6\sqrt{5}, \text{ answer.}$$

$CD = CE$
$\therefore 10\sin x = 10\cos x$
$\therefore x = 45°$
$\therefore CD = CE = 5\sqrt{2}$

Another way:



Law of Cosines in ΔACD,
$$CD^2 = b^2 + y^2 - 2by\cos\theta$$

and in ΔBCE,
$$CE^2 = a^2 + y^2 - 2ay\cos(90-\theta)$$
$$= a^2 + y^2 - 2ay\sin\theta$$

Adding,
$$CD^2 + CE^2 = a^2 + b^2 + 2y^2 - 2y(a\sin\theta + b\cos\theta)$$
$$(10\sin x)^2 + (10\cos x)^2 = a^2 + b^2 + 2y^2 - 2y\left(a\cdot\frac{a}{3y} + b\cdot\frac{b}{3y}\right)$$
$$100 = a^2 + b^2 + 2y^2 - \tfrac{2}{3}(a^2+b^2)$$
$$100 = 2y^2 + \tfrac{1}{3}(a^2+b^2)$$
$$100 = 2y^2 + \tfrac{1}{3}(3y)^2$$
$$100 = 5y^2$$

continued

15, cont'd

$$y^2 = 20$$
$$y = 2\sqrt{5}$$
$$AB = 3y = 6\sqrt{5}.$$

16. The only way 12 of the cards can total 38 is $2(13) + 2(5) + 2(1) + 6(0) = 38$.
The only way 6 additional cards can total ≥ 38 is $2(13) + 2(5) + 2(1) = 38$, so the event amounts to : no 0's in the next 6 cards.

Pr(no 0's in the next 6 cards)

$$= \frac{({}_2C_2)({}_2C_2)({}_2C_2)({}_{34}C_0)}{{}_{40}C_6}$$

$$= \frac{1}{3{,}838{,}380} \doteq .00000026$$

17. Since $\sqrt[5]{2015} \doteq 4.6$, $a \leq 4$. Note $c = \sqrt{2015 - a^5 - b^3}$.

To try $a = 1$, set \Y1 = √(2015 − A^5 − X^3) and store 1→A.
Use TABLE and observe no results are integers.
Store 2→A ; still no results are integers.
Store 3→A and observe the result X = 2, Y1 = 42.
Thus, $a = 3$, $b = 2$, $c = 42$ is a solution, and $3 \cdot 2 = 6$ is a factor of 42.
The answer is thus $42/6 = 7$.
This process can also be carried out with a calculator program.

18. Using the notation (A, C), the payoff to Anh is

- \$49 at 1 point (1, 50)
- \$48 at 2 points (1, 49), (2, 50)
- \$47 at 3 points (1, 48), (2, 49), (3, 50)

⋮

- \$1 at 49 points (1, 2), (2, 3), (3, 4), ..., (49, 50)
- \$0 at all the other points among the 50×50 points.

The average payoff is thus $[49(1) + 48(2) + \cdots + 1(49)]/50^2 = 20825/2500$
$= \$8.33$ exactly.

To find the sum, use calculator features:

sum(seq(X(50−X), X, 1, 49) gives 20825.

continued

18. (cont'd) Another way – use summation properties:

$$49(1)+48(2)+\cdots+1(49) = 1+(1+2)+(1+2+3)+\cdots+(1+2+\cdots+49)$$
$$= \frac{1\cdot 2}{2}+\frac{2\cdot 3}{2}+\frac{3\cdot 4}{2}+\cdots+\frac{49\cdot 50}{2}$$
$$= \sum_1^{49}\frac{n(n+1)}{2}$$
$$= \frac{1}{2}\left(\sum_1^{49} n^2+\sum_1^{49} n\right)$$
$$= \frac{1}{2}\left(\frac{49\cdot 50\cdot 99}{6}+\frac{49\cdot 50}{2}\right) = 20825.$$

Another way – use combinatorial properties:

$$\frac{1\cdot 2}{2}+\frac{2\cdot 3}{2}+\frac{3\cdot 4}{2}+\cdots+\frac{49\cdot 50}{2} = {}_2C_2+{}_3C_2+{}_4C_2+\cdots+{}_{50}C_2$$
$$= {}_{51}C_3 = 20825.$$

19. Playing with the numbers, we find a feasible outcome:

roll	blue	red	yellow	total
1	1	6	6	13
2	1	1	4	6
3	6	5	4	15
4	1	5	1	7

Answer $6+4+1=11$.

20. Let $x=\log_4 m=\log_6 n=\log_9(m+n)$, so:

$4^x=m$, i.e. $2^{2x}=m$
$6^x=n$, i.e. $2^x 3^x=n$
$9^x=m+n$, i.e. $3^{2x}=m+n$

$\frac{n}{m}=\left(\frac{3}{2}\right)^x$, $\quad \frac{m+n}{m}=\left(\frac{3}{2}\right)^{2x}$.

Thus $\left(\frac{3}{2}\right)^{2x}=\frac{m+n}{m}=1+\frac{n}{m}=1+\left(\frac{3}{2}\right)^x$,

so $\left(\frac{3}{2}\right)^{2x}-\left(\frac{3}{2}\right)^x-1=0$, thus $\left(\frac{3}{2}\right)^x=\frac{1+\sqrt{5}}{2}$ using Quadratic Formula.

Then $\frac{m}{n}=\frac{2}{1+\sqrt{5}}=\frac{-1+\sqrt{5}}{2}$, so $a+b+c=-1+5+2=6$.