Solutions - Fall 2019 AMATYC Student Math League Contest

1. Let d, v, K, C denote distance (in meters), velocity, Kevin, and Cara.

Then $\dfrac{v_C}{v_K} = \dfrac{2(\frac{1}{10}\text{ mile})}{2255\text{ dm.}} = \dfrac{2(.161\text{ km})}{.2255\text{ km}} = \dfrac{644}{451} = \dfrac{d_C}{d_K}$

and $d_C + d_K = 1000$

so $\dfrac{d_C}{d_K} + 1 = \dfrac{1000}{d_K}$

$\dfrac{644}{451} + 1 = \dfrac{1000}{d_K}$

$d_K = \dfrac{1000}{\frac{644}{451}+1} \doteq 412$

2. Considering the lead digits, we know $O \leq 4$.

$$\begin{array}{r} O\,N\,E \\ +\ O\,N\,E \\ \hline T\,W\,O \end{array}$$

E is either 0, 2, 4, 6, or 8.

- If E were 0, then the units digits would imply that $O = 0$, a contradiction.
- If E were 6 or 8, there would be a carry of 1 to the tens place, contradicting that W is even.
- If E were 4, then O would be 8 from the units digits, contradicting $O \leq 4$.

Thus, $E = 2$, and from the units digit, $O = 4$.
By process of elimination N is either 0, 6, or 8.

- If N were 0, then the tens digits would imply that $W = 0$, a contradiction.
- If N were 6, then the tens digits would imply that $W = 2$, contradicting that $E = 2$.

Thus, $N = 8$. (The fact that T is odd is not needed.)

$$\begin{array}{r} 4\,8\,2 \\ +\ 4\,8\,2 \\ \hline 9\,6\,4 \end{array}$$

3. Note that log is an increasing function, so the highest number has the highest log.

$$\log 2^{1000} = 1000 \log 2 \doteq 301$$

$$\log 6^{500} = 500 \log 6 \doteq 389 \ *$$

$$\log 30^{200} = 200 \log 30 \doteq 295$$

$$\log 50^{100} = 100 \log 50 \doteq 170$$

$$\log 1000^{75} = 75 \log 1000 = 225.$$

4. $$a^2 + b^2 + c^5 = 2019 \longrightarrow a = \sqrt{2019 - b^2 - c^5}$$

Note $c < 2019^{1/5} \doteq 4.58$, so c is either 1, 2, 3, or 4.

On TI-84, store $1 \to C$ and enter \Y1 = √(2019−X²−C^5)

Inspect TABLE and observe an integer result at $X = 13$, $Y1 = 43$.
Thus, $13^2 + 43^2 + 1^5 = 2019$
and $43^2 + 13^2 + 1^5 = 2019$.

The sum is $43 + 13 + 1 = 57$.

5. Since $M/3$ has remainder 2, then $M+1$ is a multiple of 3. } so $M+1 =$
" $M/5$ " " 4 " " " 5. } 15, 30, etc.
" $N/7$ " " 6 " $N+1$ " 7. } so $N+1 =$
" $N/9$ " " 8 " " " 9. } 63, 126, etc.

The smallest possible $M + N = (15-1) + (63-1) = 76$.

6. Locker Number "n" is open at the end if and only if its state is changed an odd number of times, i.e., it has an odd number of factors including 1 and n. But the factors of a number pair off with one another unless n is a square, so the answer is the number of squares between 1 and 200. Since $\sqrt{200} \doteq 14.14$, the answer is 14.

7. Directly checking, $f(2x) = 2f(x)$ is violated

by f_2 (we get $6x+2 \neq 6x+4$)
and f_3 (we get $2 \neq 4$),

whereas f_1 and f_4 satisfy both $f(ax) = af(x)$ and $f(x+y) = f(x) + f(y)$.

8. Imagine listing all such numbers in a vertical list from low to high.
In the leftmost column, all digits are 1;
in the other 7 columns, exactly half the digits are 1.

The length of the list is 2^7 numbers, so, considering place values, the contributions to the total are:

in the 10^7's place, $2^7 \times 10^7$

in the 10^6's place, $\frac{1}{2}(2^7) \times 10^6$

in the 10^5's place, $\frac{1}{2}(2^7) \times 10^5$

$\vdots \qquad\qquad\qquad \vdots$

in the 10^0's place, $\frac{1}{2}(2^7) \times 10^0$

$$\begin{aligned}
\text{total} &= 2^7(10^7) + \tfrac{1}{2}(2^7)(10^6 + \cdots + 10^0) \\
&= 20(10^6)2^6 + 2^6(1{,}111{,}111) \\
&= 2^6(20{,}000{,}000 + 1{,}111{,}111) \\
&= 2^6(21{,}111{,}111) = 1{,}351{,}111{,}104.
\end{aligned}$$

9. Notice that the range of f is $[-7, \infty)$:

$$\begin{aligned}
2x^2 + 28x + 91 &= 2(x^2 + 14x) + 91 \\
&= 2(x+7)^2 - 98 + 91 \\
&= 2(x+7)^2 - 7
\end{aligned}$$

Working backwards, first solve $f(x) = 1$:

$$\begin{aligned}
2(x+7)^2 - 7 &= 1 \\
2(x+7)^2 &= 8 \\
(x+7)^2 &= 4
\end{aligned}$$

continued

9. cont'd

$$x+7 = \pm 2$$
$$x = -5, -9$$

Since $-9 < -7$, we need to solve $f(x) = -5$:

$$2(x+7)^2 - 7 = -5$$
$$2(x+7)^2 = 2$$
$$(x+7)^2 = 1$$
$$x+7 = \pm 1$$
$$x = -6, -8$$

Since $-8 < -7$, we need to solve $f(x) = -6$:

$$2(x+7)^2 - 7 = -6$$
$$2(x+7)^2 = 1$$
$$(x+7)^2 = 1/2$$
$$x+7 = \pm\sqrt{1/2}$$
$$x = -7 \pm \sqrt{1/2}$$
$$|a-b| = (-7+\sqrt{1/2}) - (-7-\sqrt{1/2}) = 2\sqrt{1/2} = \sqrt{2}.$$

10. Denote knight, knave, and spy by T(true), F(false), and M(maybe). Arrange the options A through E next to the statements uttered:

statement	A	B	C	D	E
X says X=T	X=M	X=M	X=T	X=T	X=F
Y says X=T	Y=T	Y=F	Y=F	Y=M	Y=M
Z says Z=M	Z=F	Z=T	Z=M	Z=F	Z=T

A is false because if Y=T then Y's utterance "X=T" is true, a contradiction.

B is false because if Z=T then Z's utterance "Z=M" is true, a contradiction.

C is false because if Y=F then Y's utterance "X=T" is false, a contradiction.

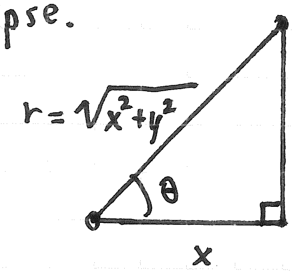
E is false because if Z=T then Z's utterance "Z=M" is true, a contradiction.

11.
$$e^{-x}\sin x - e^{-x}\cos x = 0$$
$$e^{-x}(\sin x - \cos x) = 0$$
$$\sin x - \cos x = 0$$
$$\sin x = \cos x$$
$$x = \pi/4 \text{ or } 5\pi/4 \text{ radian}$$
$$M = \pi/4 + 5\pi/4 = 3\pi/2$$
$$\csc M = \csc 3\pi/2 = -1.$$

12. Method 1 : On TI-84, press $\boxed{\text{MODE}}$ POL (polar), then $\boxed{Y=}$ $r_1 = 2((1+.5\cos(\theta))$, then $\boxed{\text{GRAPH}}$ to observe an ellipse.

Method 2 :

$$r = \frac{2}{1+0.5\cos\theta}$$



$$\sqrt{x^2+y^2} = \frac{2}{1+\frac{.5x}{\sqrt{x^2+y^2}}}$$

$$\sqrt{x^2+y^2} + .5x = 2$$

$$\sqrt{x^2+y^2} = 2 - .5x$$

$$x^2+y^2 = 4 - 2x + x^2/4$$

$$3/4\, x^2 + 2x + y^2 = 4$$

$$x^2 + \tfrac{8}{3}x + \tfrac{4}{3}y^2 = 16/3$$

$$\left(x+\tfrac{4}{3}\right)^2 + \tfrac{4}{3}y^2 = 16/3 + 16/9 \text{, an ellipse.}$$

13. When the rolling ends, the last two rolls were either:

(1,1) with probability "p"
or (1,2) with probability $1-p$.

We have $\dfrac{p}{1-p} = \dfrac{.50}{.20} = \dfrac{5}{2}$, so $p = \frac{5}{2}(1-p)$

$$\tfrac{7}{2}p = \tfrac{5}{2}$$

$$p = \tfrac{5}{7}.$$

14. We need the zeroes of $x^3 - 1/8$ that are not zeroes of $x - 1/2$

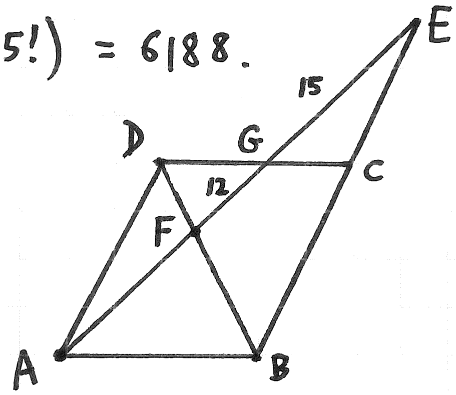
Notice, by division, that $x^3 - 1/8 = (x-1/2)(x^2+\frac{1}{2}x+\frac{1}{4})$.

The zeroes of $x^2+\frac{1}{2}x+\frac{1}{4}$ add up to $\frac{-b}{a} = -1/2$.

15. Consider a list such as (2, 4, 0, 1, 3, 2). Replace each component number with that quantity of A's, each comma with a B, and omit the parentheses: AABAAAABBABAAABAA.

This illustrates a one-to-one correspondence between the ordered lists and the "words" having 17 letters (12 A's and 5 B's).

Thus, the answer is $17!/(12!\,5!) = 6188$.



16. Since $\triangle AFD \sim \triangle EFB$,

$$\frac{AF}{27} = \frac{FD}{FB}.$$

Since $\triangle AFB \sim \triangle GFD$,

$$\frac{12}{AF} = \frac{FD}{FB}.$$

Thus, $$\frac{AF}{27} = \frac{12}{AF}$$

$$AF = \sqrt{12 \cdot 27} = 18.$$

17. At the point(s) of intersection, $f(x) = g(x)$

$$b(x-m)^2 + n = x - m$$

$$b(x-m)^2 + {}^-1(x-m) + n = 0$$

$$x - m = \frac{1 \pm \sqrt{1-4bn}}{2b}.$$

We have 2 or 1 point(s) of intersection if and only if:

$$1 - 4bn \geq 0$$

$$1 \geq 4bn$$

$bn \leq 1/4$, so the greatest possible value of bn is $\frac{1}{4}$.

18. Diagram the Math Club members with a two-by-two grid, using "x" for the percentage who are in the Physics Club but not the Quiz Team.

	Quiz	not Quiz	
Physics	P	x	$.69$
not Physics	$.79-P$	$.21-x$	$.31$
	$.79$	$.21$	

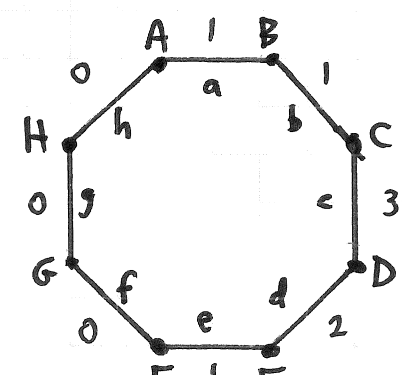
Then the only constraints are:

$$\begin{cases} P + x = .69 & \text{i.e. } P = .69 - x \\ .79 - P \geq 0 & \text{i.e. } P \leq .79 \\ .21 - x \geq 0 & \text{i.e. } x \leq .21 \\ x \geq 0 . & \end{cases}$$

To minimize P, use $x = .21$ to get $P = .69 - .21 = .48 = M$.

To maximize P, use $x = 0$ to get $P = .69 = N$.

$M + N = .48 + .69 = 1.17$



19. In the diagram, the numbers outside the octagon are the initial number of pennies, and the numbers inside are the number of pennies added just before adding the same number to the clockwise-adjacent side.
Thus, side BC ends up with $1 + a + b$ pennies,
CD " " " $3 + b + c$ " , etc.

If each side ends up with the same number of pennies, say "n", then:

$$\begin{aligned} 1 + a + b &= n & \text{Eqn } [1] \\ 3 + b + c &= n & [2] \\ 2 + c + d &= n & [3] \\ 1 + d + e &= n & [4] \\ e + f &= n & [5] \\ f + g &= n & [6] \\ g + h &= n & [7] \\ 1 + h + a &= n . & [8] \end{aligned}$$

Alternately adding and subtracting the equations, i.e., $[1] - [2] + [3] - [4] + - \cdots - [8]$,
we get: $-2 = 0$, a contradiction. So this is impossible.

20. Denote the givens by: G1: edcba has at least 3 letters placed correctly
G2: ebcda has an odd number of letters correct
G3: adcbe is not correct.

Since having exactly 4 out of 5 letters placed correctly is impossible, G1 means that edcba is correct except for 0 or 1 simple transposition. This reduces the number of candidate sequences from $_5P_5 = 120$ to $_5C_0 + {_5C_2} = 1 + 10 = 11$. We number these as follows, starting with no transposition and then transposing elements 1 and 2, 1 and 3, 1 and 4, 1 and 5, 2 and 3, 2 and 4, ..., 4 and 5:

1. edcba
2. decba
3. cdeba
4. bdcea
5. adcbe
6. ecdba
7. ebcda
8. eacbd
9. edbca
10. edabc
11. edcab

Now a direct check shows that G2 eliminates candidates 2, 4, 6, 8, 9, 11, each of which would make ebcda have exactly 2 (an even number) of letters correctly placed.

And G3 eliminates candidate 5.

That leaves only four candidates:

1. edcba
3. cdeba
7. ebcda
10. edabc

Finally, check each option A, B, C, D, and E against this list, and notice that only option C would narrow the list further, from 4 all the way to 1 candidate. So the answer is C.