Solutions – October/November 2016 AMATYC Student Mathematics League Contest

1. $$9 = \frac{10(x+5)-20}{10} = x+3, \text{ so } x = 6.$$

2.
$$\begin{array}{rl} ax + y &= c \\ x + y &= c \\ \hline (a-1)x \quad &= 0 \end{array}$$
by subtraction.

Either $a = 1$ or $x = 0$.

If $a = 1$ then the two lines are not distinct.

Thus $x = 0 = b-2$, so $b = 2$.

Since $x+y=c$, then $(b-2)+4 = c$, so $c = 4$.

Then $b+c = 2+4 = 6$.

3. Let the distances be d, so total distance $= 3d$.

Let the run, jog, and bike speeds be 9, 4.5, and x. Then the times are $\frac{d}{9}$, $\frac{d}{4.5}$, and $\frac{d}{x}$.

The overall speed is thus
$$\frac{3d}{\frac{d}{9} + \frac{d}{4.5} + \frac{d}{x}} = 7.5$$
$$3 = 7.5\left(\frac{1}{9} + \frac{1}{4.5} + \frac{1}{x}\right)$$
$$\frac{1}{x} = \frac{3}{7.5} - \frac{1}{9} - \frac{1}{4.5}$$
$$x = \left(\frac{3}{7.5} - \frac{1}{9} - \frac{1}{4.5}\right)^{-1} = 15.$$

4. In a triangle, each side is less than the sum of the other two sides.

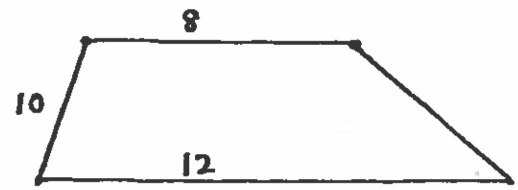
Thus, $1+2+3$ isn't feasible.

By trial and error, $2+3+4 = 9$ is the least perimeter.

5. $$(30+d)(50+d) = 8(30)(50)$$
$$d^2 + 80d - 7(1500) = 0$$
$$(d-70)(d+150) = 0$$
$$d = 70.$$

6. The side 10 is either a parallel side or not.

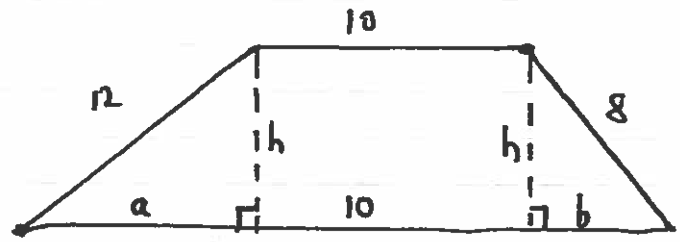
If not, then the area is:



$$A = h\left(\frac{8+12}{2}\right)\text{, which}$$

is maximized by making $h = 10$, giving $A = 10\left(\frac{8+12}{2}\right) = 100$.

If 10 is a parallel side, then the area is maximized by making it the shorter parallel side.



$$A = h\left(\frac{10 + 10 + a + b}{2}\right)$$

$$A = h\,(10 + .5a + .5b) = h\left(10 + .5\sqrt{12^2 - h^2} + .5\sqrt{8^2 - h^2}\right)$$

TI calculator $\backslash Y1 = X\left(10 + .5\sqrt{}(12^2 - X^2) + .5\sqrt{}(8^2 - X^2)\right)$

Using CALC maximum, we find that the maximum Y1 is approx. 120.816, which exceeds the 100 found earlier. So the answer is 120.

7. If person 1 is a knave, then he lied in saying "there's at least one knave", so there's no knaves – a contradiction. So person 1 is a knight.

If person 2 is a knave, then he lied in saying "there's at least two knaves", so there's exactly one knave (himself). But then person 3 must be a knight, thus telling the truth when he says "there's at least 3 knaves", a contradiction. So person 2 is a knight.

Continuing in this way, we find that the contradictions end after we establish that person 3 is a knight. With half the people knights (and half knaves), there are no contradictions, but not so if we go any further. So there are 3 knights and 3 knaves.

8. Since the roll is roughly a cylinder, then its cross-sectional area is:

$$A = \pi R^2 - \pi r^2 = \pi\left[\left(\frac{5}{2}\right)^2 - \left(\frac{7}{8}\right)^2\right] = \frac{351}{64}\pi.$$

Since the cross-section consists of 120 sheets of length 10.4 and thickness (say) "d", then we also have:

$$A = 120(10.4d).$$

Thus $120(10.4d) = \frac{351}{64}\pi$ so $d = \frac{351\pi}{64\cdot 120\cdot 10.4} \doteq 0.0138$

9. $k^2 = 2016 + 3^n$, so $k = \pm\sqrt{2016+3^n}$.

TI calculator : \Y1 = √(2016 + 3^X)

Inspecting TABLE, we find an integer output only for $X=2$, $Y_1 = 45$.
So there're two solutions $(k, n) = (45, 2)$ and $(-45, 2)$.

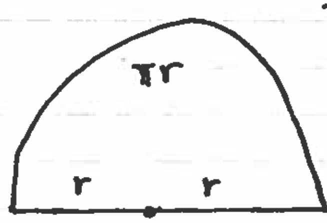
[To prove there are no solutions for $n>2$, use residues (remainders) upon division by 3:

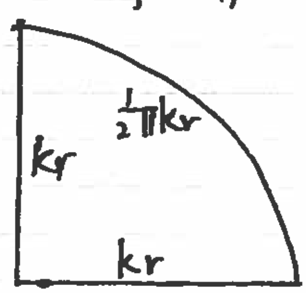
Note $k^2 = 2016 + 3^n = 3(672 + 3^{n-1}) \equiv 0 \pmod 3$, since $n>2$.
Thus $k \equiv 0 \pmod 3$, so $k = 3m$ for some integer m.

$$(3m)^2 = 2016 + 3^n$$
$$9m^2 = 9(224 + 3^{n-2})$$
$$m^2 = 224 + 3^{n-2}$$
$$m^2 \equiv 224 \equiv 2 \pmod 3, \text{ since } n>2.$$

But no square is congruent to 2 (mod 3), since $0^2 \equiv 0$, $1^2 \equiv 1$, $2^2 \equiv 1$.
Thus, there are no solutions for $n>2$.]

10. $$\pi r + 2r = 2kr + \tfrac{1}{2}\pi k r$$
$$\pi + 2 = 2k + \tfrac{1}{2}\pi k$$
$$\pi + 2 = k(2 + \pi/2)$$
$$k = \frac{\pi + 2}{\pi/2 + 2} \doteq 1.44$$





11. $P(x) = |x^2 - 28x + 160| = |(x-8)(x-20)|$.

Assuming that x is an integer,
The only way the product can be prime is if a factor $x-8$ or $x-20$ is ± 1, so we test $x = 8 \pm 1$ and $x = 20 \pm 1$:

$$\begin{aligned} P(7) &= |{-1} \cdot {-13}| \text{ is prime} \\ P(9) &= |1 \cdot {-11}| \quad " \quad " \\ P(19) &= |11 \cdot {-1}| \quad " \quad " \\ P(21) &= |13 \cdot 1| \quad " \quad " \end{aligned}$$

That gives 4 solutions, so 4 must be the correct answer since no higher option was given.

[To prove that no other solutions exist: If x is not an integer then $x = \frac{a}{b}$ with a, b integers and $b \neq 1$, and $\frac{a}{b}$ in lowest terms.
Then $P(x) = \left| \frac{a^2}{b^2} - 28 \cdot \frac{a}{b} + 160 \right|$ is an integer only if

$\frac{a^2}{b^2} - 28 \cdot \frac{a}{b} = \frac{a}{b}\left(\frac{a}{b} - 28\right)$ is an integer, which is impossible.]

12. Let abc be the larger number. Then $abc - cba = 792$,

i.e. $(100a + 10b + c) - (100c + 10b + a) = 792$

$$\begin{aligned} 99a - 99c &= 792 \\ a - c &= 8. \end{aligned}$$

The only pair of nonzero digits that differ by 8 are 9 and 1, so the numbers must be 961 and 169, whose sum is 1130.

13. $\Pr(\geq 2 \text{ people born on same date}) = 1 - \Pr(\text{all dates differ})$

$$= 1 - \frac{30}{30} \times \frac{29}{30} \times \frac{28}{30} \times \cdots \times \frac{30-n+1}{30}$$

$$\text{or} \quad 1 - \frac{30 \; {}_nP_r \; n}{30^n}.$$

By trial and error, or using the TABLE on a calculator, we find that the smallest n that makes the above expression exceed 50% is 7.

14. a) $x^2 + 15xy = x + 15y \longrightarrow x^2 - x = 15y - 15xy$
$$x(x-1) = {}^-15y(x-1)$$
so either $x=1$ or $x = {}^-15y$

b) $y^2 - xy = 15x + y$: If $x=1$ then
$$y^2 - y = 15 + y$$
$$y^2 - 2y - 15 = 0$$
$$(y-5)(y+3) = 0$$
$$y = 5 \text{ or } {}^-3 \quad \text{(2 solutions)}$$

: If $x = {}^-15y$ then
$$y^2 + 15y^2 = -15^2 y + y$$
$$16y^2 = -224y$$
$$y^2 = -14y$$
so either $y=0$ or $y = -14$

If $y=0$ then $x = {}^-15(0) = 0$ (1 solution)

If $y = {}^-14$ then $x = {}^-15({}^-14) = 210$ (1 solution)

So there is a total of 4 solutions $(x,y) = (1,5), (1,{}^-3), (0,0), (210,{}^-14)$.

15. Original series: $S = a + \frac{2}{3}a + \frac{4}{9}a + \cdots = \frac{a}{1-\frac{2}{3}}$
$$\therefore \tfrac{1}{3}S = a.$$

Adjusted series: $T = 2\underbrace{\left(a + \frac{4}{9}a + \frac{16}{81}a + \cdots\right)}_{\text{geometric}} + \frac{1}{2}\underbrace{\left(\frac{2}{3}a + \frac{8}{27}a + \frac{32}{243}a + \cdots\right)}_{\text{geometric}}$

$$T = 2\left(\frac{a}{1-\frac{4}{9}}\right) + \frac{1}{2}\left(\frac{\frac{2}{3}a}{1-\frac{4}{9}}\right)$$

$$T = \frac{18}{5}a + \frac{3}{5}a = \frac{21}{5}a = \frac{21}{5}\left(\frac{1}{3}S\right) = \frac{7}{5}S = 1.4S$$

16. Let r_n = the number of rankings of n players, with ties allowed. Clearly $r_1 = 1$, and $r_0 = 1$ (the empty set), and $r_2 = 3$.

To find r_3, note that deleting the first-place player(s) leaves a ranking of 0, 1, or 2 players. Thus, $r_3 = \binom{3}{3}r_0 + \binom{3}{2}r_1 + \binom{3}{1}r_2$
$$r_3 = 1 \cdot 1 + 3 \cdot 1 + 3 \cdot 3 = 13.$$

- continued -

16. (continued) Similarly, $r_4 = \binom{4}{4} r_0 + \binom{4}{3} r_1 + \binom{4}{2} r_2 + \binom{4}{1} r_3$

$$r_4 = 1 \cdot 1 + 4 \cdot 1 + 6 \cdot 3 + 4 \cdot 13 = 75,$$

and $r_5 = \binom{5}{5} r_0 + \binom{5}{4} r_1 + \binom{5}{3} r_2 + \binom{5}{2} r_3 + \binom{5}{1} r_4$

$$r_5 = 1 \cdot 1 + 5 \cdot 1 + 10 \cdot 3 + 10 \cdot 13 + 5 \cdot 75 = 541.$$

17. Notice that subtracting the first from the second equation gives
$x + 2y + 3z = 6$.
Adding this to the second equation gives $7x + 7y + 7z = 14$,
i.e. $x + y + z = 2$.

Another way: in matrix form, the two equations become:

$$\begin{bmatrix} 5 & 3 & 1 & 2 \\ 6 & 5 & 4 & 8 \end{bmatrix}.$$

Reducing this to row-echelon form (by hand or on a calculator, e.g. MATRIX MATH rref on TI-84) gives:

$$\begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 4 \end{bmatrix}.$$

Adding these rows together gives $[1 \ 1 \ 1 \ 2]$, i.e., $x+y+z=2$.

18. $abc_{15} = 1abc_6$

$$c + 15b + 225a = c + 6b + 36a + 216$$
$$189a + 9b = 216$$
$$a = \frac{216 - 9b}{189}$$

TI calculator $\backslash Y1 = (216 - 9X)/189$
Use TABLE to look for integer outputs: $X = 3$, $Y1 = 1$.
So $a + b = 1 + 3 = 4$.

19. For A, the winning sequences of green (G) and red (R) chips are:
G, GRG, GRGRG, GRGRGRG, etc.
Since $Pr(G) = 4/12 = 1/3$ and $Pr(R) = 8/12 = 2/3$,
we have $Pr(\text{A wins}) = \frac{1}{3} + \frac{1}{3} \cdot \frac{2}{3} \cdot \frac{1}{3} + \frac{1}{3} \cdot \frac{2}{3} \cdot \frac{1}{3} \cdot \frac{2}{3} \cdot \frac{1}{3} + \cdots$.

This geometric series with common ratio $2/9$ sums to

$$\frac{1/3}{1 - 2/9} = \frac{3}{7}.$$

20. $\begin{cases} a^2+b^2+c^2+d^2 = 2500 \longrightarrow & c^2+d^2 = 2500-a^2-b^2 \\ (a+50)(b+50) = cd \longrightarrow & 2cd = 2(a+50)(b+50) \end{cases}$

Subtract: $(c-d)^2 = -[a^2+b^2+2ab+100(a+b)+2500]$

$$(c-d)^2 = -[(a+b)^2+100(a+b)+2500]$$

$$(c-d)^2 = -(a+b+50)^2.$$

Since these two squares are additive inverses, they must both be 0.

Thus, $c = d$ and $a+b+50=0$,

i.e., $b+50 = -a$.

This simplifies $(a+50)(b+50)=cd$ to $(a+50)(-a) = c^2$,

i.e., $\pm\sqrt{(a+50)(-a)} = c$.

TI calculator $\backslash Y1 = \sqrt{(^-X(X+50))}$

Use TABLE to look for integer outputs:

$X=a$	$Y1=\pm c$
0	0
−1	7
−5	15
−10	20
−18	24
−25	25
−32	24
−40	20
−45	15
−49	7
−50	0

Due to the $\pm c$, we double the number of solutions here, except for the 0's:

$2(9)+2 = 20$ solutions.