Solutions — Fall 2018 AMATYC Student Math League Contest

1. $5\#2 = \dfrac{5\cdot 4 + 2\cdot 25}{5 + 4\cdot 25} = \dfrac{70}{105} = \dfrac{2}{3}$

$2\#(5\#2) = 2\#\frac{2}{3} = \dfrac{2\cdot\frac{4}{9} + \frac{2}{3}\cdot 4}{5 + \frac{4}{9}\cdot 4} = \dfrac{32/9}{61/9} = \dfrac{32}{61}$.

2. By symmetry of the definition, $x\#y = y\#x$, so the operation is commutative.

To test associativity: $1\#(1\#2) = 1\#\frac{2}{3} = \frac{10}{49}$;

$(1\#1)\#2 = \frac{1}{3}\#2 = \frac{14}{13} \neq \frac{10}{49}$,

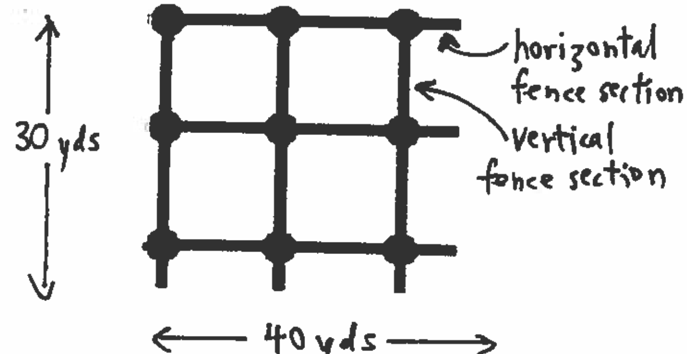
so the operation is not associative.

3. There are 31 rows of 41 posts each, so $P = 41 \times 31 = 1271$.

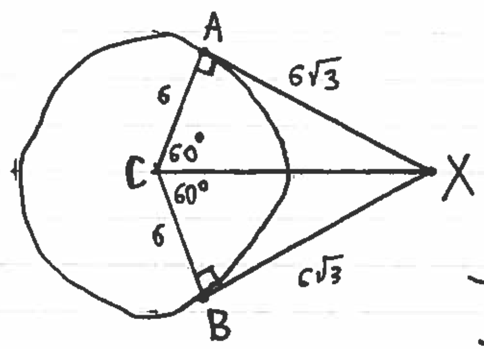
For the fence sections, there are:
- 31 stretches of 40 horizontal ones
- 30 sets of 41 vertical ones,

so $F = 31\cdot 40 + 30\cdot 41 = 2470$

$P + F = 1271 + 2470 = 3741$.



4.



Since both triangles are 30°-60°-90°, then $AX = BX = 6\sqrt{3}$.
The triangles re-arranged make a rectangle that's 6 by $6\sqrt{3}$, so the area is $36\sqrt{3}$.

5. For convenience, re-write the problem as an addition:

$$\begin{array}{r} \text{E I G H T} \\ +\ \ \text{N I N E} \\ \hline \text{S E V E N} \\ \scriptstyle 5\ 4\ 3\ 2\ 1 \end{array} \longleftarrow \text{and label the columns.}$$

From column 5, we know $E + 1 = S$ and there is a "carry" from column 4.

continued

5. (continued) Therefore, from column 4 we have:

$I + N = E + 9$ or $E + 10$, depending whether there is a "carry" from column 3.

Similarly, from column 1 we have:

$T + E = N$ or $N + 10$.

Combining these two facts leads to: $I + T = 9$ or 10.

Similarly, comparing columns 1 and 2, we have:

Either $T + E = N$ and $H + N = E$ or $E + 10$,

or $T + E = N + 10$ and $H + N + 1 = E$ or $E + 10$.

Combining these two facts leads to: $T + H = 10$ or 9, respectively.

This means that if we guess, say, I, then T and H are constrained:

I	T	H	
8	1	9	options where $I + T = 9$ and $T + H = 10$
7	2	8	
⋮			
8	2	7	options where $I + T = 10$ and $T + H = 9$.
7	3	6	
⋮			

Substituting each triplet above, in turn, and playing with the options each time, we soon arrive at two solutions:

$$\begin{array}{r} 47382 \\ +\ \ 6764 \\ \hline 54146 \end{array} \qquad \begin{array}{r} 16873 \\ +\ \ 4641 \\ \hline 21514 \end{array}$$

Thus, the possible values of N sum to $6 + 4 = 10$.

6. Since $x = b^2$ has roughly twice as many digits as b, then b has roughly one-third of the 91 digits.
   - But if b has 30 digits then x^2 has 59 or 60 digits, and $30 + 60 < 91$.
   - And if b has 31 digits then x^2 has 61 or 62 digits, and $31 + 61 > 91$.

Thus, the outcome is impossible; its probability is 0.

7. Simple enumeration method:

Since 8 out of 10 digits are used, 2 digits are unused: either $\{1,2\}$ or $\{1,3\}$ or $\{1,4\}$ or $\cdots$ or $\{6,6\}$, 18 possibilities.

- If $\{1,2\}$ are unused, that leaves $\{3,4,4,5,5,6,6,6\}$. The number of possible strings of these 8 digits would be $8!$ if the digits were distinct, but taking repeats into account gives us $8!/(2!2!3!) = 1680$ different ones.

- Similarly for $\{1,3\}$ unused, we get $8!/(2!2!3!) = 1680$,
  and for $\{1,4\}$ unused, we get $8!/(2!3!) = 3360$.

- Continuing in this way for all 18 subsets, we get:
  $$3(1680) + 8(3360) + 3(5040) + 1(6720) + 3(10{,}080) = 84{,}000.$$

Grouping method:

If we use all 10 digits, the number of distinct 10-digit strings is $10!/(2!2!3!) = 151{,}200$. We finish by grouping these into subsets according to their first 8 digits: in each group, only the last 2 digits vary from string to string.

If the last 2 digits are equal to each other, the possibilities are:

• 44,	which leaves	$\{1,2,3,5,5,6,6,6\}$,	thus $8!/(2!3!)$	$= 3360$ strings
• 55,	" "	$\{1,2,3,4,4,6,6,6\}$,	" "	3360
• 66,	" "	$\{1,2,3,4,4,5,5,6\}$,	" $8!/(2!2!)$	$= 10{,}080$
			total	16,800

If the last 2 digits are not equal to each other (such as 12, 21, 35, or 53), the number of possibilities is the complement, $151{,}200 - 16{,}800$. Since every such pair of digits has 2 permutations (such as 12 and 21), these cases pair off: in each pair, the first 8 digits are the same in one member of the pair as in the other.

Therefore, the answer is $16{,}800 + \frac{1}{2}(151{,}200 - 16{,}800) = 84{,}000.$

8. Since $2018^{1/6} \doteq 3.55$, we know that $a = 1, 2,$ or 3.

And $b^2 = 2018 - a^6 - c^3$
$b = \sqrt{2018 - a^6 - c^3}$.

On TI-84 calculator:
store $\backslash Y1 = \sqrt{(2018 - A\hat{}6 - X\hat{}3)}$
and $1 \rightarrow A$.
Examine TABLE to find integer outcomes: $(X, Y1) = (12, 17)$ only.

Store $2 \rightarrow A$ and find integer outcomes: $(9, 35)$ only.

Store $3 \rightarrow A$ and find integer outcomes: $(4, 35)$ and $(10, 17)$ only.

Thus, there are 4 distinct solutions.

9. Denote knight, knave, spy by T (true), F (false), M (maybe).
Summarize the 3 statements and 5 choices in a chart:

statement	A	B	C	D	E
X: Z = F	M	M	T	T	F
Y: X = T	T	F	F	M	T
Z: Z = M	F	T	M	F	M

In columns A, B, C, E, the circled fact is inconsistent with one of the statements. This leaves only D as plausible.

10. $$x^2 + y = xy + x$$
$$x^2 - xy + y - x = 0$$

$$x(x-y) + 1(x-y) = 0$$
$$(x-1)(x-y) = 0$$
$x - 1 = 0$ or $x - y = 0$
$x = 1$ or $x = y$, so the graph is two lines.
(vertical line) ($45°$ line)

11. By direct inspection,

- the snacks eaten on the first cycle are $1, 3, 5, \ldots, 125$ (increment $= 2$)
- " " " second " $4, 8, 12, \ldots, 124$ ($= 4$)
- " " " third " $6, 14, 22, \ldots, 118$ ($= 8$)
- " " " fourth " $2, 18, 34, \ldots, 114$ ($= 16$)
- " " " fifth " $10, 42, 74, 106$ ($= 32$)
- " " " sixth " $26, 90$ ($= 64$)
- " " " seventh " 58,

Which leaves only snack 122 for the dog.

12. $(\log_8 x^2)(\log_x 8)^2 = 1$; or on TI-84 calculator:

enter \Y1 = ln(X²)/ln(8)*(ln(8)/ln(X))²

Use 2^{nd} CALC zero to find the root, $X = 64$.

$(2\log_8 x)(\log_x 8)^2 = 1$

$\dfrac{2\log_8 x}{(\log_8 x)^2} = 1$

$2 = \log_8 x$

$8^2 = x$, so $50 < x \leq 100$.

13. Since $5!$ includes factors 3 and 5, all factorials $5!$ or larger have remainder 0 when divided by 15.

Thus, $R =$ remainder in $(1! + 2! + 3! + 4!)/15$
$= (1 + 2 + 6 + 24)/15$
$= 33/15 = 2$ remainder 3, so $R = 3$.

Note that 2^2 is the square of an integer, so $N = 2$, and $R + N = 5$.

14. Note that $p = 1/A$ and $q = 1/B$, so $1/A + 1/B = 14$.

Since $(-9, 10)$ is a point on the line, $-9A + 10B = 1$, i.e., $B = .1 + .9A$

Substituting this result gives $1/A + 1/(.1 + .9A) = 14$,

which simplifies to $0 = 126A^2 - 5A - 1 = (9A - 1)(14A + 1)$.

Thus, either $A = -1/14$ and $B = .1 + .9A = 1/28$ and $p = -14$ and $q = 28$, contradicting the negative slope;
or $A = 1/9$ and $B = .1 + .9A = .2$ so $A + B = 1/9 + .2 = 14/45$.

15. $0 = ax^2+bx+c = a(x-r)(x-s) = a\left[x^2+(-r-s)x+rs\right]$,

so $b = a(-r-s)$ and $c = ars$, i.e., $r+s = \frac{-b}{a}$ and $rs = \frac{c}{a}$.

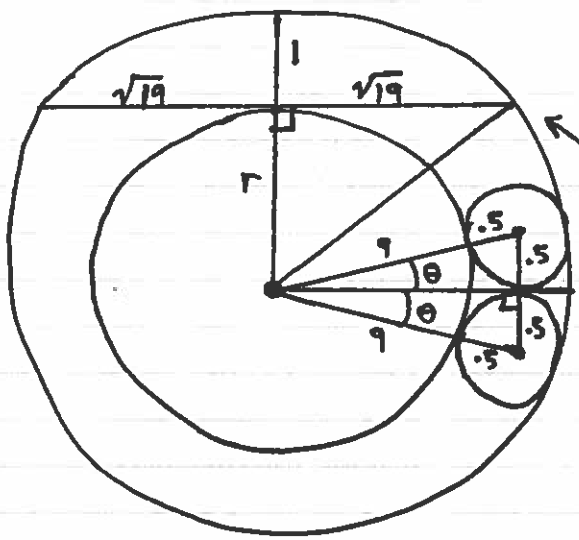
Similarly $0 = x^2+dx+e = \left(x-\frac{r}{1+r}\right)\left(x-\frac{s}{1+s}\right)$,

so $d = \frac{-r}{1+r} - \frac{s}{1+s}$ and $e = \frac{rs}{(1+r)(1+s)}$.

$$\therefore\ d+e = \frac{-r(1+s)+-s(1+r)+rs}{(1+r)(1+s)} = \frac{-r+-s+-rs}{1+r+s+rs}$$

$$= -\left(\frac{r+s+rs}{1+r+s+rs}\right)$$

$$= -\left(\frac{-\frac{b}{a}+\frac{c}{a}}{1+\frac{-b}{a}+\frac{c}{a}}\right) = \frac{b-c}{a-b+c}.$$

16. 

Let R, r be the radii of the large and small concentric circles, respectively.

By the Pythagorean Theorem,

$$r^2+(\sqrt{19})^2 = R^2 = (r+1)^2$$
$$19 = 2r+1$$
$$r = 9.$$

Then for every consecutive pair of small circles, the central angle 2θ between their centers satisfies:

$$\sin\theta = \frac{.5}{9+.5} = \frac{1}{19};$$

using radian measure, the number of small circles than can fit is:

$$\frac{2\pi}{2\theta} = \frac{2\pi}{2\sin^{-1}(1/19)} \doteq 59.66, \text{ so the answer is } 59.$$

17. Since 1111 has prime factors 11 and 101, then $P = 101$.

The parabola has equation of form $y = a(x-2-\sqrt{3})(x-2+\sqrt{3})$

$y = a\left[(x-2)^2-3\right] = a(x^2-4x+1)$

$y = ax^2-4ax+a$, so $y_{int} = a = -3$.

The Vertex has x-coördinate midway between $2+\sqrt{3}$ and $2-\sqrt{3}$, i.e. $x=2$,

so its y-coördinate is $K = -3(x^2-4x+1) = -3(-3) = 9$.

Thus $P+K = 101+9 = 110$.

18. In any three consecutive terms of an arithmetic sequence,

$$2(\text{middle term}) = \text{first term} + \text{last term}$$
$$2(4x+2) = (8x-1) + (2x-6)$$
$$4 = 2x-7$$
$$x = 11/2.$$

Thus, the sum of the three terms is:

$$(\text{first} + \text{last}) + \text{middle} = 2(\text{middle}) + \text{middle}$$
$$= 3(\text{middle})$$
$$= 3(4x+2)$$
$$= 3(4 \cdot 11/2 + 2) = 72.$$

19. Let d denote the one-way distance in miles.
The times for the outward and return trips, respectively, are:

$$\frac{d \text{ miles}}{60 \text{ miles/hr}} = \frac{d}{60} \text{ hour}, \quad \frac{d \text{ miles}}{20 \text{ miles/hr}} = \frac{d}{20} \text{ hour}.$$

Thus, the overall speed is:

$$\frac{2d \text{ miles}}{\frac{d}{60} + \frac{d}{20} \text{ hour}} = \frac{2d}{4d/60} \frac{\text{miles}}{\text{hour}} = 30 \text{ mph}.$$

20. For P_1: There are 6 ways to reach sum $= 5$, namely $(1,1,3)$, $(1,3,1)$, $(3,1,1)$, $(1,2,2)$, $(2,1,2)$, $(2,2,1)$.
Each of these has probability $\frac{1}{6} \times \frac{1}{6} \times \frac{1}{6}$, so $P_1 = 6 \times \frac{1}{6} \times \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$.

For P_2: The total probability must be 100%,

$$\text{so } p_1 + p_2 + p_3 + p_4 + p_5 + p_6 = 1.00$$

$$p_1 + p_1 + 2p_1 + 3p_1 + 4(2p_1) + 3 \cdot 4(2p_1) = 1$$
$$39 p_1 = 1$$

Thus, $P_2 = p_2 = p_1 = 1/39$.

For P_3: Since $999/39 \doteq 25.6$, there are 25 multiples of 39 between 1 and 999.

Thus, $P_3 = 25/999$.

Ordering the results, we have $\frac{1}{36} > \frac{1}{39} > \frac{25}{999}$

; i.e., $P_1 > P_2 > P_3$.