SML Round 2 2012-2013
Short Answers

1. **D** 13 + 12 + 13 = 38.
2. **D** $8.1-1.4 < c < 8.1+1.4 \Rightarrow 6.7 < c < 9.5$. Since c must be even, it must be 8.
3. **E** The matrix $\begin{bmatrix} \pi & \pi+e & \pi+2e \\ \pi+3e & \pi+4e & \pi+5e \end{bmatrix}$ reduces to $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$. So $b-a = 2-(-1) = 3$.

All

4. **C** 2014: prime factors are 2, 19, 53. Reduce each by 1: 1, 18, 52. No
   2015: prime factors are 5, 13, 31. Reduce each by 1: 4, 12, 30. No
   2016: prime factors are 2, 3, 7. Reduce each by 1: 1, 2, 6. Yes!
5. **B** Since the two lines intersect at a point (a,0), solve the system of equations:
   $\begin{matrix} 2a+b=0 \\ ma-6=0 \end{matrix}$ to get $mb+12=0$.
6. **B** If n = 1, there is only 1 solution, (2,2). If n = 2, there are 2 solutions, (4,4) and (6,3). If n =3, there are 2 solutions, (6,6) and (12,4). When n =4, there are 3 solutions, (8,8), (12,6), and (20,5).
7. **B** If b = 5, use a graphing calculator's TABLE feature to look for integer values of y for $y=\sqrt{1988-x^3}$ when x is also an integer. There are none. For b = 10, do the same with $y=\sqrt{1913-x^3}$. There is an integer solution when x = 4 and y = 43. So, $a+b+c=4+10+43=57$.

8. **A** Since there are 7 prime factors (3,3,5,5,7,11,11) and 6 letters in AMATYC, then there are 5 possible values for M, T, Y, and C. Thus, three of {M,T,Y,C} must be prime (in particular, one of them must be 7) and one of them must be a product of two primes. Since the values assigned to each letter must be less than 27, the product of primes must be 15. This means A must be assigned 11 and {M,T,Y,C} = {3,5,7,15}. So, M + T + Y + C = 3 + 5 + 7 + 15 = 30.

9. **B** Let $P(x)=ax^3+bx^2+cx+d$. Then $P(0)=d$. Since $P(0)\cdot P(3)=139$ and 139 is prime, $P(3)=1$ or 139. All coefficients are nonnegative integers, so $P(3)=139$ and thus $P(0)=1=d$. $P(1)\cdot P(2)=689=13\cdot 53$. Again, all coefficients are nonnegative integers, so $P(2)=53$ and $P(1)=13$. Solve the system
   $\begin{Bmatrix} 27a+9b+3c=138 \\ 8a+4b+2c=52 \\ a+b+c=12 \end{Bmatrix}$ to get $a=3$, $b=5$, and $c=4$. Then $P(x)=3x^3+5x^2+4x+1$
   and $P(-1)=-3+5-4+1=-1$.

10. **6.175** $\cos(t)=\cos(2k\pi-t)=\cos(2k\pi+t)$. $t^\circ=\dfrac{t\pi}{180}$ radians.

$\underline{\cos(t)=\cos(2k\pi+t)}$: $t=2k\pi+\dfrac{\pi t}{180}\Rightarrow t=\dfrac{2k\pi}{1-\dfrac{\pi}{180}}$. When k = 1, t = 6.395. Larger values of k give larger values of t.

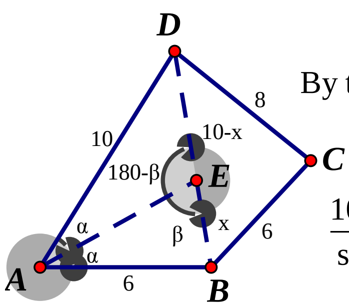
$\dfrac{\pi t}{180}=2k\pi+t\Rightarrow t=\dfrac{2k\pi}{\dfrac{\pi}{180}-1}$. All values are negative.

$\underline{\cos(t)=\cos(2k\pi-t)}$: $t=2k\pi-\dfrac{\pi t}{180}\Rightarrow t=\dfrac{2k\pi}{1+\dfrac{\pi}{180}}$. When k = 1, t = 6.175. Larger values of k give larger values of t.

$\dfrac{\pi t}{180}=2k\pi-t\Rightarrow t=\dfrac{2k\pi}{\dfrac{\pi}{180}+1}$. When k = 1, t = 6.175 and all other values are larger.

So the smallest such value for t is 6.175.

11. **A** Draw a picture.



By the Pythagorean Theorem, DE = 10. Using the Law of Sines and the diagram shown, $\dfrac{x}{\sin\alpha}=\dfrac{6}{\sin\beta}$ from triangle AEB and $\dfrac{10-x}{\sin\alpha}=\dfrac{10}{\sin(180-\beta)}=\dfrac{10}{\sin\beta}$ from triangle AED. Solve each for $\sin\beta$ to get $\dfrac{6\sin\alpha}{x}=\dfrac{10\sin\alpha}{10-x}$. Solve for x to get BD = $\dfrac{15}{4}$.

12. **C** Line L has the points (0,2) and (4,0) or (2,0) and (0,4). So the possible equations for line L are $y=-\dfrac{1}{2}x+2$ and $y=-2x+4$. Line M has the points (0,4) and (6,0) or (4,0) and (0,6). The possible equations for line M are $y=-\dfrac{2}{3}x+4$ and $y=-\dfrac{3}{2}x+6$. Finding the possible intersections of lines L and M gives (12,-4), (4,0), (0,4), and (-4,12), with values of 3a + b equal to 32, 12, 4, and 0 respectively. The only possibility listed that is not a possible value of 3a + b is 8.

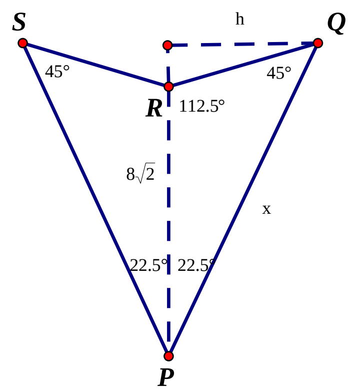
13. **B** Let n be the number of trips taken and let a_k be the length of trip k. $a_k=a_1+2(k-1)$.

$$366=\sum_{k=1}^{n}(a_1+2(k-1))=n^2+na_1-n=n(n+a_1-1)=n^2+n(a_1-1).$$

$n^2+n(a_1-1)-366=0$. Keeping in mind that $a_1<91$, the only possible factorization is

$(n+61)(n-6)=0$. Then $n = 6$ and thus $a_1 = 56$. The lengths of the trips are thus 56, 58, 60, 62, 64, and 66 days. The only one in the given list is 58.

14. The question was unclear – all students were marked as having this one correct.
15. **E** Draw a picture:



Area = 2(Area of triangle PQR) = $2\left(\frac{1}{2}\cdot 8\sqrt{2}\cdot h\right)=8\sqrt{2}\cdot h$

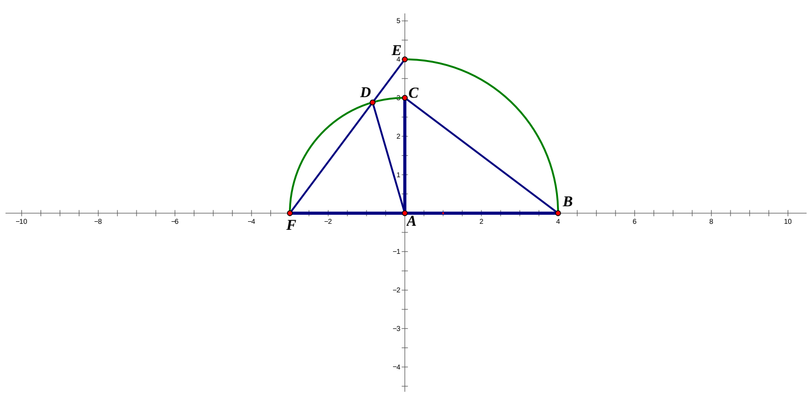
$\frac{h}{x}=\sin 22.5^\circ \Rightarrow h = x\sin 22.5^\circ$

By the Law of Sines, $\frac{x}{\sin 112.5^\circ}=\frac{8\sqrt{2}}{\sin 45^\circ}$ so $x=\frac{8\sqrt{2}\sin 112.5^\circ}{\sin 45^\circ}$

Area = $8\sqrt{2}\left(\frac{8\sqrt{2}\sin 112.5^\circ}{\sin 45^\circ}\right)\sin 22.5^\circ = 64$

16. **C** $a^2 = 2b^2+2 \Rightarrow b^2=\frac{a^2-2}{2}$. Since b must be positive, $b=\sqrt{\frac{a^2-2}{2}}$. Use the TABLE feature of a graphing calculator with x and y replacing a and b, respectively, and look for integer values in each column. $a - b = 58 - 41 = 17$.
17. **B** There are 5! = 120 5-digit numbers. The numbers containing the digits 1, 2, 3, 4, and 5 for which adjacent digits are not consecutive numbers are 13524, 14253, 24135, 24153, 25314, 31425, 31524, 35241, 35142, 41352, 42531, 42513, 52413, and 53142. Thus the probability is $\frac{14}{120}=\frac{7}{60}$.

18. **B** Again, draw a picture:



The region we need to find the area of consists of the quarter circle in QI, sector ADF, and triangle AED.

The circle $x^2+y^2=9$ and the line $y=\frac{4}{3}x+4$ intersect at the point $D=\left(-\frac{21}{25},\frac{72}{25}\right)$.

The area of the quarter circle in QI is 4π. If θ is the angle in sector ADF, the area of the sector is $\frac{1}{2}r^2\theta = \frac{1}{2}(3^2)\sin^{-1}(24/25) \approx 5.7915$ (Note: $\frac{72}{25} = 3\sin\theta$)

The area of triangle AED = $\frac{1}{2}\cdot 4\cdot\frac{21}{25} = 1.68$. So, total area $\approx 4\pi + 5.7915 + 1.68 \approx 20.04$

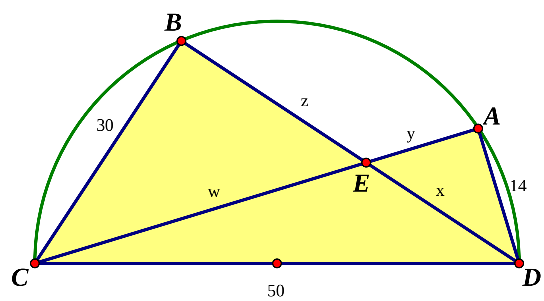
19. **D** Let $x^2 + mx + n = (x+a)(x+b)$. Then $a + b = m$ and $ab = n$.

Let $x^2 + mx - n = \left(x - \frac{a}{c}\right)(x + bc)$. Again, $ac = n$ and this time $bc - \frac{a}{c} = m$.

Then $bc - \frac{a}{c} = a + b \Rightarrow \frac{b}{a} = \frac{c+1}{c(c-1)}$. Solutions can be obtained using any rational value for c (other than 0 or 1).

When $c = 2$, $\frac{b}{a} = \frac{3}{2}$. Corresponding pairs (m,n) are (6,5), (24,10), (54,15), and (96,20). (Check those out!). When $c = 3$, the same solutions are obtained. $c = 4$ generates (60,17) $c = 5$ generates (30, 13), $c = 6$ generates no solutions (recall that $n < 100$), $c = 7$ generates (84,25). All other rational values of c either generate one of the solutions already given, or give results for which $n > 100$. So there are 7 such pairs.

20. **D** Draw a picture:



Because CD is the diameter of a circle, the angles at A and B are right angles. Using the Pythagorean Theorem, $BC = 30$ and $AC = 48$. From the diagram, $w + y = 48$ and $x + z = 40$.

Triangles ACD and BCD are similar, so $\frac{14}{30} = \frac{7}{15} = \frac{x}{w} = \frac{y}{z}$. So $xz = wy$.

Then, $(40-z)z = (48-y)y \Rightarrow (40-z)z = \left(48 - \frac{7}{15}z\right)\cdot\frac{7}{15}z \Rightarrow 15z(40-z) = 7z\left(48 - \frac{7}{15}z\right)$

Solving for z yields $z = 0$ (impossible) or $z = 22.5$. Then $x = 17.5$, $y = 10.5$, and $w = 37.5$.

Shaded area = area of triangle CDA + area of triangle CDB – area of triangle CDE
= 336 + 600 – area of triangle CDE

By Heron's Formula, area of CDE = $\sqrt{52.5(52.5-17.5)(52.5-37.5)(52.5-50)} = 262.5$

So, Polygon area = 336 + 600 – 262.5 = 673.5.