Solutions – Winter/Spring 2019 AMATYC Student Math League Contest

1. $16{,}800 \text{ mm} \times \frac{1 \text{ cm.}}{10 \text{ mm.}} \times \frac{1 \text{ in.}}{2.54 \text{ cm.}} \times \frac{1 \text{ ft.}}{12 \text{ in.}} \doteq 55.12 \text{ ft} \longrightarrow 56 \text{ ft.}$

2. Any set A with "n" elements has 2^n subsets, including itself and the empty set, or 2^{n-1} proper subsets. So $M = 2^4 - 1 = 15$.

   Among $\{1, 2, 3, 4\}$, the distinct pairwise differences are $-3, -2, -1, 1, 2, 3$, so $N = 6$. Thus $M+N = 21$.

3. If 2 \$20 bills are used, then the remaining \$10 can be returned in $1+3=4$ ways, as shown.

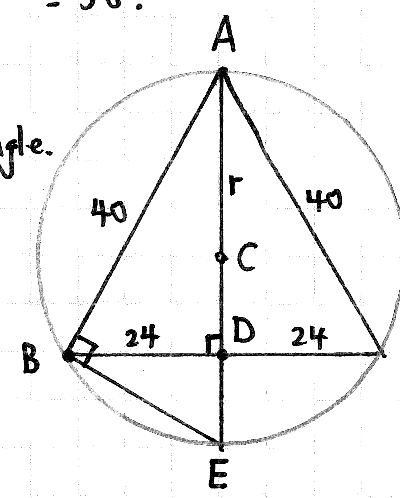
\$10	\$5	\$1
1	0	0
0	2	0
0	1	5
0	0	10

   Similarly, if 1 \$20 bill is used, then the remaining \$30 can be returned in $1+3+5+7 = 16$ ways.

   Finally, if no \$20 bill is used, then the \$50 can be returned in $1+3+5+\cdots+11 = 36$ ways.

   So the answer is $4+16+36 = 2^2+4^2+6^2 = 56$.

4. In the figure, $AD = \sqrt{40^2 - 24^2} = 32$, i.e., $\triangle BDA$ is an expanded 3-4-5 right triangle.



   But $\triangle EBA$ is a right triangle similar to $\triangle BDA$, so:

$$\frac{2r}{40} = \frac{40}{32} \text{ or } \frac{5}{4}$$

$$r = \frac{5}{4} \cdot 20 = 25.$$

5. $2019 = 3 \cdot 673$ is the prime factorization.

Since $2^9 < 673 < 2^{10}$, $M = 10$.

Since $16^2 < 2019 < 16^3$, $N = 3$.

Thus, $M - N = 10 - 3 = 7$.

6. The sides have squared lengths as follows:

$a^2 = (8-6)^2 + (8-4)^2 = 20$

$b^2 = (8-10)^2 + (8-7) = 5$

$c^2 = (6-10)^2 + (4-7)^2 = 25 = a^2 + b^2$, so it's a right triangle.

The legs are altitudes of length $\sqrt{20}$ and $\sqrt{5}$, respectively.
The area of the triangle is $\frac{1}{2} bh = \frac{1}{2}\sqrt{20}\sqrt{5} = 5$.

Therefore if the third altitude, with length "H", is constructed on the hypotenuse of length $\sqrt{25} = 5$, then the area is again $5 = \frac{1}{2} BH = \frac{1}{2} \cdot 5H$,
so $H = 2$.

The answer is thus $\sqrt{20} + \sqrt{5} + 2 \doteq 8.7$

7. $2x^3 + x^2 + cx + d = (x+1)(2x^2 + ax + b)$.

Comparing the coëfficients of x^2 on both sides of the equation, we get:

$$1 = 2 + a$$
$$a = -1.$$

The non-real roots are those of the quadratic,

$$x_1 = \frac{-a + \sqrt{a^2 - 8b}}{4}, \quad x_2 = \frac{-a - \sqrt{a^2 - 8b}}{4},$$

whose sum is $x_1 + x_2 = -\frac{1}{2} a = \frac{-1}{2}(-1) = \frac{1}{2}$.

(The given fact $d + c = 29$ is unneeded.)

8. For $n = 3, 4, 5, \ldots$ we have $n^{n-1} = 3^2$, 4^3 or 8^2, 5^4 or 25^2, ...

Thus, $N = 6$, since 6^5 is not the square of an integer.

By the Rational Roots Theorem, as "b", "c", and "d" are varied,

the rational roots of $5x^4 + bx^3 + cx^2 + dx + 6$

are $\pm 6, \pm 3, \pm 2, \pm 1, \pm \frac{1}{5}, \pm \frac{2}{5}, \pm \frac{3}{5}, \pm \frac{6}{5}$.

The product of the positive ones is $(36)\left(\frac{36}{25^2}\right) = \left(\frac{36}{25}\right)^2$.

Similarly for the negative ones, so the answer is $\left(\frac{36}{25}\right)^4 \doteq 4.30$

9. This chart summarizes the five choices, denoting knight, knave, spy with T, F, M (true, false, maybe):

	A.	B.	C.	D.	E.	
X	M	M	T	T	F	X says "X=M": this rules out C and D.
Y	T	F	F	M	M	Y says "X=M is true": this rules out B.
Z	F	T	M	F	T	Z says "Z≠M": this rules out A and D.

Thus, the correct choice is E.

10. Each row of the chart below assumes that the checked (✓) signal is correct, and shows the number of errors in the other signals, where F is the first signal.

A.	B.	C.	D.	F.	
✓	4	1	3	2	
4	✓	3	3	2	(duplicate 3's)
1	3	✓	2	3	(duplicate 3's)
3	3	2	✓	3	(duplicate 3's)

Each row except the top has duplicate numbers of errors, so only the top row meets the given requirement, and signal A must be correct.

11. Starting at upper right, label each square with a number indicating how many permissible ways there are to enter that square.

- If a square has a neighbor only above it or to the right, then the label remains the same.
- If a square has neighbors both above it and to the right, then the label is the sum of the two inputs.

1	1	1	1	●
3	2	1	■	1
3	■	2	1	1
5	2	2	■	1
10	5	3	1	1

The final label, 10, is the answer.

12. Method A : Denoting $c = \cos t$, we have:

$$\cos 2t = 2\cos^2 t - 1 = 2c^2 - 1$$

$$\cos 4t = 2\cos^2 2t - 1 = 2(2c^2-1)^2 - 1 .$$

Continuing in this way, it's clear that $\cos 2t$, $\cos 4t$, $\cos 8t, \dots$ have only even powers of c, so the coëfficient of c^3 is 0.

Method B : Let $\cos 8t = a_0 + a_1 \cos t + a_2 \cos^2 t + a_3 \cos^3 t + \cdots$

Then $\cos 8(\pi - t) = a_0 + a_1 \cos(\pi - t) + a_2 \cos^2(\pi - t) + a_3 \cos^3(\pi - t) + \cdots$

Use $\cos(\pi - t) = -\cos t$:

$$\cos(8\pi - 8t) = a_0 - a_1 \cos t + a_2 \cos^2 t - a_3 \cos^3 t + - \cdots$$

Use $\cos(2n\pi + A) = \cos A$:

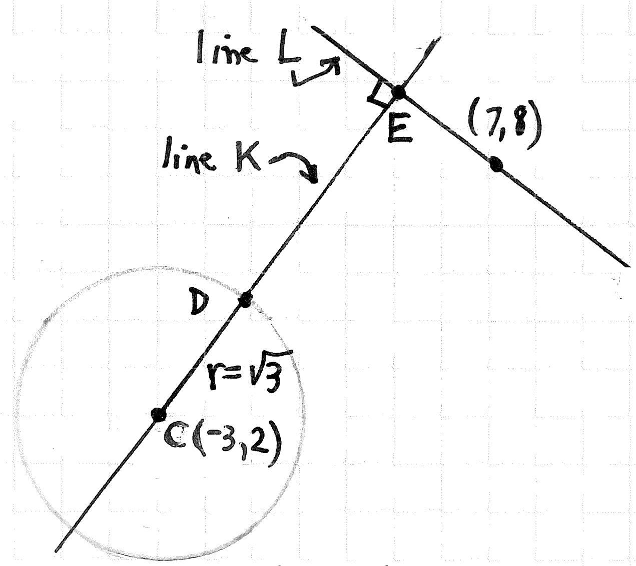
$$\cos(-8t) = a_0 - a_1 \cos t + a_2 \cos^2 t - a_3 \cos^3 t + - \cdots$$

Use $\cos(-A) = \cos A$:

$$\cos 8t = a_0 - a_1 \cos t + a_2 \cos^2 t - a_3 \cos^3 t + - \cdots$$

Comparing this with $\cos 8t$ above, we get $a_1 = 0$, $a_3 = 0$, $a_5 = 0, \dots$

13. Method A



The line "L" through (7,8) parallel to $\langle 3,-4\rangle$ has slope $-4/3$, so its equation is:

$$y = -\frac{4}{3}x + b$$
$$8 = -\frac{4}{3}(7) + b$$
$$b = \frac{52}{3}$$
$$L:\ y = -\frac{4}{3}x + \frac{52}{3}.$$

The shortest line segment from L to the given circle is perpendicular to L (thus has slope $3/4$) and passes through the center C(-3,2). Thus, it lies on a line "K" with equation:

$$y = \frac{3}{4}x + b$$
$$2 = \frac{3}{4}(-3) + b$$
$$b = \frac{17}{4}$$
$$K:\ y = \frac{3}{4}x + \frac{17}{4}.$$

The point of intersection "E" of L and K can be found:

$$y = -\frac{4}{3}x + \frac{52}{3} = \frac{3}{4}x + \frac{17}{4}$$
$$\frac{157}{12} = \frac{25}{12}x$$
$$x = 157/25 = 6.28 \text{ exactly}$$
$$y = \frac{3}{4}(6.28) + \frac{17}{4} = 8.96 \text{ exactly}$$

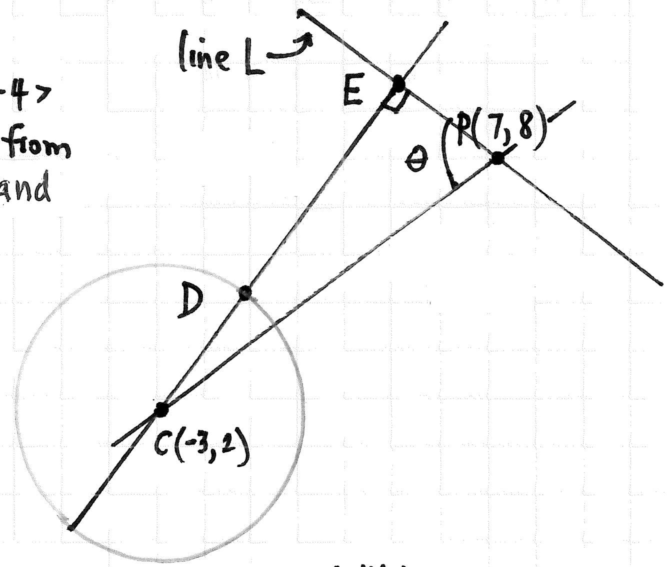
The distance from C(-3,2) to E(6.28, 8.96) is

$$\sqrt{(6.28+3)^2 + (8.96-2)^2} = 11.6 \text{ exactly},$$

so the answer is the distance $DE = CE - r = 11.6 - \sqrt{3} \doteq 9.9$

13. Method B

The line "L" through $P(7,8)$ parallel to $\langle 3,-4 \rangle$ has slope $-4/3$. The shortest line segment from L to the given circle is perpendicular to L and passes through the center $C(-3,2)$.



The line $\overleftrightarrow{PC}$ has slope $\dfrac{8-2}{7--3} = \dfrac{6}{10} = \dfrac{3}{5}$.

In right triangle ΔPCE, the angle θ at vertex P can be found using a trig identity, where each tangent represents a slope:

$$\tan(A-B) = \frac{\tan A - \tan B}{1+\tan A \tan B} = \frac{3/5 - (-4/3)}{1 + 3/5 \cdot -4/3}$$

$$\tan\theta = \frac{29/15}{3/15} = \frac{29}{3}, \text{ so } \theta = \tan^{-1}\frac{29}{3}.$$

Thus,

$CE = (CP)(\sin\theta)$ where $CP = \sqrt{(8-2)^2+(7--3)^2} = \sqrt{136}$

$CE = \sqrt{136}\,\sin\left(\tan^{-1}\frac{29}{3}\right)$

$CE = 11.6$ using calculator

$DE = CE - r = 11.6 - \sqrt{3} \doteq 9.9$

14. $a^2 + b^2 = 2019^2$

$b^2 = 2019^2 - a^2$

$b = \sqrt{2019^2 - a^2}$

For $a = 1, 2, 3, \ldots$ test whether $\sqrt{2019^2 - a^2}$ is an integer:

On TI-84+:

```
PRGM : TEST
: For (A, 1, 2018)
: If fPart(√(2019²- A²)) = 0
: Disp N
: End
```

The output is 1155 and 1656, so the answer is $1155 + 1656 = 2811$.

15. Method A: Notice that $5(\text{Eqn } 1) + (\text{Eqn } 2) = 2(\text{Eqn } 3)$.
Thus, Eqn 3 is redundant. Notice that Eqn 1 and Eqn 2 are not parallel, so they meet in a line.

Method B: The matrix representation is:

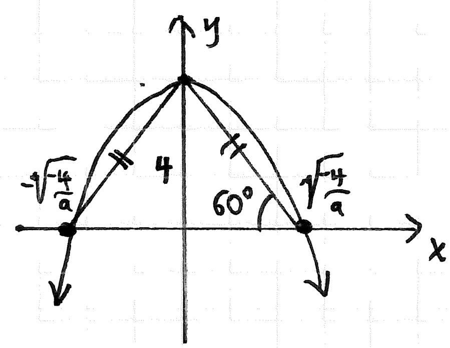
$$\left[\begin{array}{ccc|c} 2 & -6 & -8 & 15 \\ -8 & -8 & 6 & -65 \\ 1 & -19 & -17 & 5 \end{array}\right].$$

Reduce the matrix by Gaussian elimination (by hand or on a calculator) and notice that exactly one row became all 0's, and no row became all 0's except in the final column. Thus, the graph is the intersection of two planes that meet in a line.

16. Symmetry about the y-axis means f is an even function, so "b" (the coëfficient of an odd power of x) must be 0. Since max $f = 4$, then $c = 4$.

The other two intercepts of $f(x) = ax^2 + 4$ satisfy:

$$f(x) = 0 \longrightarrow 0 = ax^2 + 4 \longrightarrow x = \pm\sqrt{-4/a}.$$



Thus, $\tan 60° = 4/\sqrt{-4/a}$

$$\sqrt{3} = 4/\sqrt{-4/a}$$

$$3 = 16/(-4/a)$$

$$a = -3/4$$

$$a + b + c = -3/4 + 0 + 4 = 13/4.$$

17. $g = \ln e^x = x$ is cts and 1:1 ✓
$h = x^2$ (if $x \geq 0$) and $-x^2$ (if $x \leq 0$) is cts and 1:1 ✓
$k = x^2$ is not 1:1
$m = 1/(x+1)$ is not cts at $x = -1$
$n = x/(x^2+1)$ is not 1:1 (the graph violates the Horizontal Line Property)
$p = \sin x$ is periodic, thus not 1:1
$q = \tan^{-1} x$ is cts and 1:1 (observe the graph) ✓
$r = x/(|x|+1)$ is cts and 1:1 (observe the graph) ✓

4 meet the requirements

18. $\sum_{n=1}^{100} a_n = \sum_{n=1}^{100}[m+(n-1)d] = 100m + d\sum_{n=1}^{100}(n-1)$

$= 100m + d\,\frac{99\cdot 100}{2} = 100m + 4950d.$

$\sum_{n=1}^{10} g_n = \sum_{n=1}^{10} k\cdot 2^{n-1} = k\sum_{n=1}^{10} 2^{n-1} = k(2^{10}-1) = 1023k.$

Thus,

$$100m + 4950d = 1023k$$
$$100m = 1023k - 4950d$$
$$100m = (3\cdot 11\cdot 31)k - (2\cdot 3^2\cdot 5^2\cdot 11)d$$
$$100m = 3\cdot 11(31k - 150d)$$

Since 100 is not divisible by 3 or 11, then "m" must be divisible by 33.

19. Let L = distance hiked on level ground before turnaround (in miles)
U = " " " uphill " " "
D = " " " downhill " " "

On each segment, time = distance ÷ speed,
so for the time before turnaround we have $\frac{L}{4} + \frac{U}{3} + \frac{D}{6}$.
After the turnaround, the segments that were uphill become downhill, and vice-versa, so the time after turnaround is $\frac{L}{4} + \frac{D}{3} + \frac{U}{6}$.
The total time was 8 hours $= \left(\frac{L}{4} + \frac{U}{3} + \frac{D}{6}\right) + \left(\frac{L}{4} + \frac{D}{3} + \frac{U}{6}\right)$,

so $8 = \frac{L}{2} + \frac{U}{2} + \frac{D}{2}$

$16 = L + U + D$,

and the total miles hiked was thus 32 miles.

20. Using a 6×6 grid, recall that the totals on a pair of fair dice have probabilities:

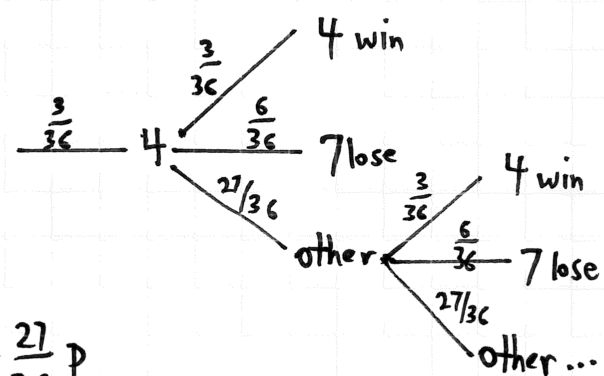
$$Pr(2) = Pr(12) = 1/36, \quad Pr(3) = Pr(11) = 2/36, \quad \ldots, \quad Pr(7) = 6/36.$$

A win occurs on roll 1 with probability: $Pr(7 \text{ or } 11) = \frac{6}{36} + \frac{2}{36} = \frac{2}{9}$.

In the case where roll 1 was a 4, the probability was $3/36$. Following this, on all subsequent rolls the possible outcomes are:

$$Pr(4) = 3/36, \quad Pr(7) = 6/36, \quad Pr(\text{other}) = 1 - \frac{3}{36} - \frac{6}{36} = \frac{27}{36}.$$

Thus, after a 4 on roll 1, the chance of an eventual win is:



$$\frac{3}{36} + \frac{27}{36}\left[\frac{3}{36} + \frac{27}{36}\left(\frac{3}{36} + \cdots\right.\right.,$$

or denoting this by "p", we have $p = \frac{3}{36} + \frac{27}{36}p$

$$\frac{9}{36}p = \frac{3}{36}$$

$$p = \frac{3}{9}.$$

Similarly, for the case of 5 we get $\frac{4}{36} + \frac{26}{36}\left[\frac{4}{36} + \frac{26}{36}\left(\frac{4}{36} + \cdots\right.\right.$,

which leads to $p = \frac{4}{10}$.

In the case of 6, we get $p = \frac{5}{11}$.

By symmetry, the cases 8, 9, and 10 match those of 6, 5, and 4, respectively.

Thus, the total win probability is:

$$\underbrace{\frac{2}{9}}_{\text{roll 1} = 7 \text{ or } 11} + \left(\underbrace{\frac{3}{36}\cdot\frac{3}{9}}_{\text{or } 4} + \underbrace{\frac{4}{36}\cdot\frac{4}{10}}_{\text{or } 5} + \underbrace{\frac{5}{36}\cdot\frac{5}{11}}_{\text{or } 6} + \underbrace{\frac{5}{36}\cdot\frac{5}{11}}_{\text{or } 8} + \underbrace{\frac{4}{36}\cdot\frac{4}{10}}_{\text{or } 9} + \underbrace{\frac{3}{36}\cdot\frac{3}{9}}_{\text{or } 10}\right)$$

$$= \frac{2}{9} + 2\left(\frac{3}{36}\cdot\frac{3}{9} + \frac{4}{36}\cdot\frac{4}{10} + \frac{5}{36}\cdot\frac{5}{11}\right) = \frac{244}{495}.$$