SML Round 1 2013-2014
Short Answers

1. **C** $\dfrac{1\,Euro}{\$1.25 \text{ x } 1.10} \text{ x } \$2.20 = 1.6\,Euros$
2. **D** Solve the system $\begin{array}{r} 2x - \ \ y = \ \ a \\ -2x + 2y = 2b \end{array}$ for y to get $y = a + 2b$
3. **B** $\log(\log(\log(10^{(10^{10})}))) = \log(\log(10^{10})) = \log(10) = 1$
4. **B** 7,777 = 6,611 + 1,166; 77,777 = 29,488 + 48,289; 777,777 = 666,111 + 111,666; 7,777,777 = 2,948,888 + 4,828,889. There is no such sum for 777.
5. **D** Line L has slope –*A*/*B* and line M has slope –*C*/*A*. Since these lines are perpendicular, (–*A*/*B*)(–*C*/*A*) = -1 and so *C* = -*B*. From the sum of the equations, we get *A* + *C* = 6 and *A* + *B* = 10. So *B* – *C* = 4. Then 2*B* = 4, or *B* = 2. So *C* = -2 and *A* = 8. The slopes of lines L and M are thus -4 and ¼.
6. **B** In $\begin{array}{r} 4M4 \\ -\underline{TYC} \\ SML \end{array}$ either *Y* =0 and *C* = 1,2, or 3 or *Y* = 9 and *C* =5,6,7, or 8. A little trial and error yields $\begin{array}{r} 434 \\ -\underline{196} \\ 238 \end{array}$ in which *M* = 3.
7. **A** Suppose the numbers are x_1, x_2, x_3, x_4, x_5 in order. Since the minimum equals the median, $x_1 = x_3$ and so $x_2 = x_1$ as well. Also, the range equals the minimum, so $x_5 - x_1 = x_1$ or $x_5 = 2x_1$. The missing number must be the mean, so $\dfrac{x_1 + x_1 + x_1 + x_4 + 2x_1}{5} = x_4$, or $x_4 = \dfrac{5}{4}x_1$. The ratio of maximum to mean is then $\dfrac{2x_1}{\frac{5}{4}x_1} = \dfrac{8}{5} = 1.6$
8. **D** The slope of the line through the given points is $-\dfrac{3}{2}$ so the slope of line *L* is $\dfrac{2}{3}$ (since *L* must be the perpendicular bisector of the line through the given points) Line *L* goes through (4,7), the midpoint of the line segment joining the two given points. So *L* has equation $y - 7 = \dfrac{2}{3}(x - 4)$, or $3y - 2x = 13$.
9. **D** Since $\log_a b$ and $\log_b a$ are reciprocals, $(\log_8 x^2)(\log_x 8)^2 = (2\log_8 x)(\log_x 8)(\log_x 8) = 2\log_x 8 = 1$ implies $\log_x 8 = \dfrac{1}{2}$ and thus $x^{1/2} = 8$ or *x* = 64.
10. **C** If Al is a knight, then Bo is a knight and Cy is a knave. This works! If Al is a knave, then Bo is a knave and Cy is a knight, which doesn't work.

11. **B** Time to number crunch! Let $c=\sqrt{2013-a^4-2b^2}$. For a = 1, there are no positive integer solutions for c. Trying a = 2, 3, and 4 and there are no positive integer solutions. When a = 5, b = 26 and c = 6 work. So a + b + c = 37.

12. **E** Each term is the average of the terms that came before, so all of the terms from a_3 on are the same (check this out!) and are equal to the average of a_1 and a_2. Then $12(a_1+a_2)=N\left(\frac{a_1+a_2}{2}\right)$. So $12=\frac{N}{2}$ or $N=24$.

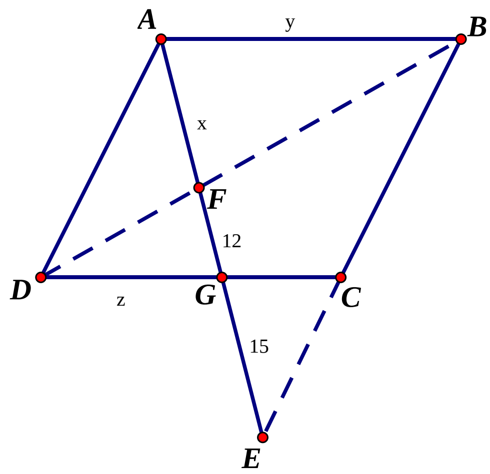
13. **E** $x^2=4y^2+81 \Rightarrow x^2-4y^2=81 \Rightarrow (x-2y)(x+2y)=81=3^4$. This leads to 10 systems of equations for finding x and y, all of which have integer solutions:

$x-2y$	1	-1	3	-3	9	-9	27	-27	81	-81
$x+2y$	81	-81	27	-27	9	-9	3	-3	1	-1
(x,y)	(41,20)	(-41,-20)	(15,6)	(-15,-6)	(9,0)	(-9,0)	(15,-6)	(-15,6)	(41,-20)	(-41,20)

14. **8** On a graphing calculator, let $y_1=abs(k-abs(abs(x)-6))-2$ and look for the number of zeroes as you vary the value of k. When k =8, there are 5 zeroes.

15. **A** There are n - 1 fractions in the cluster of fractions with denominator n. The number of fractions in the list up to the end of the cluster with denominator n is $1+2+3+\cdots+n=\frac{n(n+1)}{2}$. The largest value of n for which $\frac{n(n+1)}{2}\le 2013$ is n = 62. $\frac{62\cdot 63}{2}=1953$. So, there are 1953 fractions in the list beginning at $\frac{1}{2}$ and ending at $\frac{62}{63}$. We need 2013-1953=60 more fractions in the next cluster with denominator 64. So the 2013th element of the sequence is $\frac{60}{64}$ and 60 + 64 = 124.

16. **B** Let $AF = x$, $AB = y$, and $DG = z$ as shown:



ΔADG is similar to ΔECG, so $\dfrac{15}{y-z} = \dfrac{x+12}{z}$. Also, ΔABF is similar to ΔGDF, so $\dfrac{12}{z} = \dfrac{x}{y}$. This implies that $y = \dfrac{xz}{12}$. Putting this into the first equation gives $\dfrac{15}{\frac{xz}{12} - z} = \dfrac{x+12}{z}$.

This yields $324z = x^2 z$ and, since $z \neq 0$, we get $324 = x^2$, or $x = 18$.

17. **A** If the first 3 flips are Heads, then Ha wins. $P(\text{Ha wins}) = \left(\dfrac{1}{2}\right)^3 = \dfrac{1}{8}$. If one of the first 3 flips is Tails (i.e., if Ha doesn't win), then Mo can win by choosing THH, which will appear before HHH.

18. **E** From the discussion for #17, the probability that Mo wins is $1 - \dfrac{1}{8} = \dfrac{7}{8}$.

19. **A** Let $P(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$. If a rational number $\dfrac{m}{n}$ (in lowest terms) is a root, then $P\left(\dfrac{m}{n}\right) = a_4\dfrac{m^4}{n^4} + a_3\dfrac{m^3}{n^3} + a_2\dfrac{m^2}{n^2} + a_1\dfrac{m}{n} + a_0 = 0$ or

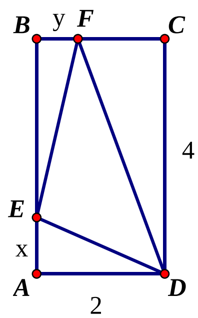
$a_4m^4 + a_3m^3n + a_2m^2n^2 + a_1mn^3 + a_0n^4 = 0$. Then $a_4m^4 + a_3m^3n + a_2m^2n^2 + a_1mn^3 = -a_0n^4$ **(1)**

Case 1: n is even. Since $\dfrac{m}{n}$ is in lowest terms, m must be odd. But then since all coefficients of P are odd, the left side of **(1)** is odd and the right side is even. This is impossible.

Case 2: n is odd. If m is even, then the left side of **(1)** is even and the right side is odd. On

the other hand, if m is odd, then the left side of **(1)** is even and the right side is odd. This is also impossible.
So, no rational number can be a root of P.

20. **B** Let $x = \overline{AE}$ and let $y = \overline{BF}$ as shown:



Since the areas of the three outer triangles are equal, $x = \frac{1}{2}y(4-x) = \frac{1}{2}(4)(2-y)$. The area of the rectangle is 8, so the area of ΔDEF is $8-3x$. $x = \frac{1}{2}(4)(2-y) = 4-2y$ so $y = \frac{4-x}{2}$.
Then $x = \frac{1}{2}\left(\frac{4-x}{2}\right)(4-x)$ or $x^2 - 12x + 16 = 0$. Thus $x = 6 \pm 2\sqrt{5}$. $6+2\sqrt{5}$ is too large, so $x = 6-2\sqrt{5}$. So the ratio of the area of triangle DEF to the area of triangle ADE is
$$\frac{8-3(6-2\sqrt{5})}{6-2\sqrt{5}} = \frac{-10+6\sqrt{5}}{6-2\sqrt{5}} = \sqrt{5}$$