Solutions – Winter 2017 AMATYC Student Mathematics League Contest

1. Among the possible perimeters are

$$\begin{aligned} 2+3+4 &= 9 \\ 2+4+5 &= 11 \\ 2+5+6 &= 13 \\ 3+4+6 &= 13 \end{aligned} \Big\} \text{ same perimeter}$$

Among the impermissible outcomes are $1+4+5$ (not a triangle)
$1+2+2$ (isosceles).

If equilateral triangles are considered isosceles, the answer is 13 (see above). However, it was discovered that by some definitions, equilateral is not considered isosceles. This would allow solutions such as
$3+3+3 = 2+3+4 = 9$.
Therefore, this problem was ruled ambiguous, and every response was considered as correct.

2. $$45 \text{ gleeks} = 45 \text{ gleeks} \times \frac{3 \text{ zeffs}}{7 \text{ gleeks}} \times \frac{2 \text{ gems}}{5 \text{ zeffs}} = \frac{54}{7} \text{ gems} < 8 \text{ gems, yes.}$$

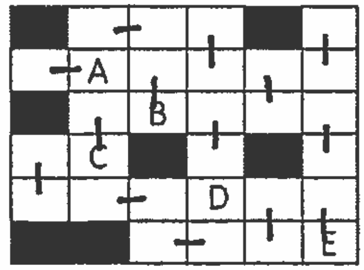
$$\begin{aligned} \text{Change} &= \left(8 - \frac{54}{7}\right) \text{gem} = \frac{2}{7} \text{ gem} \\ &= \frac{2}{7} \text{ gem} \times \frac{5 \text{ zeffs}}{2 \text{ gems}} \times \frac{7 \text{ gleeks}}{3 \text{ zeffs}} \times \frac{11 \text{ zorks}}{15 \text{ gleeks}} \\ &= \frac{11}{9} \text{ zork} \doteq 1.2 \text{ zork} \end{aligned}$$

3. $$\begin{aligned} a^2 + b^2 + 2ab + 16a + 16b &= 36 \\ (a+b)^2 + 16(a+b) &= 36 \\ 4\left(\frac{a+b}{2}\right)^2 + 32\left(\frac{a+b}{2}\right) &= 36 \\ 4c^2 + 32c - 36 &= 0 \\ c^2 + 8c - 9 &= 0 \\ (c+9)(c-1) &= 0 \\ c &= -9, 1 \\ -9+1 &= -8 \text{, answer.} \end{aligned}$$

4. Since $\$29/5$ passengers $< \$41/7$ passengers, we want to use lots of cars but without empty seats.
The optimal is

$$\begin{aligned} &7 \text{ cars} \longrightarrow 35 \text{ passengers @ } \$203 \\ \text{and } &2 \text{ vans} \longrightarrow \underline{14 \text{ passengers @ } \$82} \\ & \qquad\qquad\; 49 \text{ passengers @ } \$285. \end{aligned}$$

5. By trial and error, we find the solution D, as shown in the diagram here, where each of the 14 dashes represents a domino covering 2 unit squares.



6. $$x^2 + xy + x + 3y = 6$$

$$(x^2 + x - 6) + (xy + 3y) = 0$$

$$(x+3)(x-2) + (x+3)y = 0$$

$$(x+3)(x-2+y) = 0$$

$x + 3 = 0$ or $x - 2 + y = 0$, the union of two lines (one vertical and one oblique, thus intersecting)

7. Method (i). From the choices, we can assume a, b are between 1 and 29. Create a program to test each option, such as for TI-84:

```
:For( A, 1, 29)
:For ( B, 1, 29)
:If (A+4√(B))^3⁻¹+(A-4√(B))^3⁻¹ =3
:Disp A, B
:End
:End
```

Running this displays the output 9, 5 so the answer is $9 + 5 = 14$.

Method (ii) Let $c = a + b$. With a calculator, table the values of the function, such as for TI-84:

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\Y1 = (X+4√(C-X))^3⁻¹ + (X-4√(C-X))^3⁻¹ -3 .
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We need values $Y1 = 0$. From the choices, we can assume $c = 14$ or 18 or etc.

Store $14 \to C$ and examine TBL :

we find $Y1 = 0$ at $X = 9$, so the answer is $c = 14$.

continued

7. (cont'd) Method (iii) $\sqrt[3]{a+4\sqrt{b}} + \sqrt[3]{a-\sqrt{b}} = 3$. Cube both sides.

$$(a+4\sqrt{b}) + 3\left[(a+4\sqrt{b})^2(a-4\sqrt{b})\right]^{1/3} + 3\left[(a+4\sqrt{b})(a-4\sqrt{b})^2\right]^{1/3} + (a-4\sqrt{b}) = 27$$

$$2a + 3\left[(a+4\sqrt{b})(a-4\sqrt{b})\right]^{1/3}\left[\underbrace{(a+4\sqrt{b})^{1/3}+(a-4\sqrt{b})^{1/3}}_{=3 \text{ by given}}\right] = 27$$

$$2a + 9\left[a^2-16b\right]^{1/3} = 27$$

$$a^2-16b = \left(3-\tfrac{2}{9}a\right)^3$$

Since a, b are integers, a is divisible by 9.
Substituting, we get:

$a = 0 \longrightarrow b = -27/16$, not integer
$a = 9 \longrightarrow b = 5$
$a = 18 \longrightarrow b = 325/16$, not integer.
$\vdots$

Since $a+b$ is increasing, this is convincing that the smallest possible result is $a+b = 9+5 = 14$.

8. $$\begin{pmatrix} a & 8 \\ -3 & b \end{pmatrix}\begin{pmatrix} a & 8 \\ -3 & b \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} a^2-24 & 8(a+b) \\ -3(a+b) & b^2-24 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$ so $a^2-24=1$, $b^2-24=1$, $a+b=0$

$a = \pm 5$, $b = \pm 5$, $a = -b$
$(a, b) = (5, -5)$ or $(-5, 5)$
$|a-b| = 10$

9. $11 = f(4) - f(2) = (16+4b+c) - (4+2b+c) = 12+2b$, so $b = -1/2$

Thus $f(4) - f(0) = (16+4b+c) - (0+0+c) = 16+4b = 14$.

10. Denote knight, knave, and spy by T, F, and M (true, false, and maybe).
By guided trial-and-error, a feasible option for choice A is
$(X, Y, Z) = (T, M, F)$.
Similarly for choice B : $(X, Y, Z) = (T, M, F)$
C : (F, M, T)
E : (M, F, T)
Only choice D is infeasible.

11. $S(n) = 1 + 2 + 3 + \cdots + n = \dfrac{n(n+1)}{2}$.

Method (i) $2S(n) = n(n+1)$, so we must check whether a string of 2's is factorable into an "n" and $n+1$.

Since $2016(2017) = 4{,}066{,}272$ it is sufficient to check $2, 22, 222, \cdots, 2222222$.

For example, to check whether $222 = n(n+1)$ for any n, we note that "n" and $n+1$ would be consecutive integers sandwiching $\sqrt{222} = 14.9$, so the only possibility is $14 \times 15 = 210 \neq 222$, which fails.

Doing this, we find that the only successful outcome is $2 = 1 \times 2$, proving that there is only 1 solution.

Method (ii) Table the values with a calculator, such as for TI-84:

\Y1 = X(X+1)/2

Using TBL we see no values other than $S(1)=1$ that have all digits 1, suggesting that there is only 1 solution.

Method (iii) $111\cdots1 = 1 + 10 + 10^2 + \cdots + 10^m$ for $m \geq 0$

Required: $\dfrac{n(n+1)}{2} = \dfrac{10^{m+1} - 1}{10 - 1}$

$$\frac{9}{2}n(n+1) + 1 = 10^{m+1} \quad \text{for } m+1 \geq 1$$

$$9n^2 + 9n + 2(1 - 10^{m+1}) = 0$$

Using the Quadratic Formula,

$$n = \frac{-3 + \sqrt{8 \cdot 10^{m+1} + 1}}{6} \quad \text{so } 8 \cdot 10^{m+1} + 1 \text{ is a perfect square.}$$

For $n = 2017$, we get $\frac{9}{2}2017(2017) + 1 = 10^{m+1}$

$$m+1 = \log_{10} 18{,}316{,}378 \doteq 7.26$$

\Y1 = √(8*10^X+1)

Using TBL, we see no integer values (for $m+1 = X$ between 1 and 7) except for $X=1$, proving that there is only 1 solution.

12. Let x = capacity of radiator in quarts

$$\tfrac{1}{2}\left(\tfrac{2}{3}x+4\right)+11 = x$$

$$\tfrac{1}{3}x + 13 = x$$

$$13 = \tfrac{2}{3}x$$

$$x = \tfrac{39}{2} \text{ quarts} = \tfrac{39}{8} \text{ gallons} = 4.875$$

13. In both cases, $P(\text{win}) = P(\text{"yes"}) \cdot P(\text{win, given "yes"}) + P(\text{"no"}) \cdot P(\text{win, given "no"})$.

Thus, $P(M) = (.01)(1) + (.99)(1/99) = .02$

and $P(T) = (.5)(1/50) + (.5)(1/50) = .02$

so $P(M)/P(T) = 1$

14. Denote $a = 3+5i$, $\bar{a} = 3-5i$, $b = 4-7i$, $\bar{b} = 4+7i$.

Then $P(x) = c\,(x-12)(x-a)(x-\bar{a})(x-b)(x-\bar{b})$.

$$P(0) = c \cdot (-12)(a \cdot \bar{a})(b \cdot \bar{b}) = -53{,}040$$

$$-12c\,\|a\|^2\,\|b\|^2 = -53{,}040$$

$$-12c\,(3^2+5^2)(4^2+7^2) = -53{,}040$$

$$-12c\,(34)(65) = -53{,}040$$

$$c = 2$$

so $P(x) = 2\,(x-12)(x-a)(x-\bar{a})(x-b)(x-\bar{b})$, and

the coefficient of x^3 is 2(sum of all 2-fold products among roots)

$$= 2\left[(12a+12\bar{a}) + (12b+12\bar{b}) + a\bar{a} + b\bar{b} + (ab+\bar{a}b+a\bar{b}+\bar{a}\bar{b})\right]$$

$$= 2\left[12(a+\bar{a}) + 12(b+\bar{b}) + a\bar{a}+b\bar{b} + (a+\bar{a})(b+\bar{b})\right]$$

$$= 2\left[12 \cdot 2\operatorname{Re}(a) + 12 \cdot 2\operatorname{Re}(b) + \|a\|^2 + \|b\|^2 + 2\operatorname{Re}(a) \cdot 2\operatorname{Re}(b)\right]$$

$$= 2\left[12 \cdot 6 + 12 \cdot 8 + 34 + 65 + 6 \cdot 8\right]$$

$$= 630.$$

15. $\text{Area} = 1^2 - \text{area(unshaded)}$

$= 1 - \text{area(4 petals)}$

$= 1 - \text{area(8 half-petals)}$

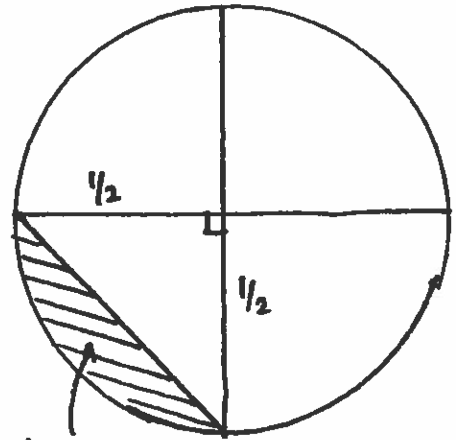
$= 1 - 8 \cdot \text{area(half-petal)}$

$= 1 - 8\left[\frac{1}{4}\pi r^2 - \frac{1}{2} bh\right]$

$= 1 - 8\left[\frac{1}{4}\pi\left(\frac{1}{2}\right)^2 - \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\right]$

$= 1 - 8\left[\frac{1}{16}\pi - \frac{1}{8}\right]$

$= 2 - \pi/2 \doteq 0.45$



half-petal = quarter-circle minus right triangle

16. Since $999/7 \doteq 142.7$, then 142 of the integers are divisible by 7

$999/11 \doteq 90.8$ " 90 " " " " " " 11

$999/77 \doteq 12.97$ " 12 " " " " " " 7 <u>and</u> 11

Thus, the number of integers divisible by exactly one of 7 or 11 is

$$142 + 90 - 2(12) = 208.$$

17. $(1+1)+(1+2)+(1+3)+\dots+(1+n) = 1(n) + (1+2+\dots+n)$

$= n + \frac{1}{2}n(n+1)$

$= \frac{1}{2}n^2 + \frac{3}{2}n$

Very roughly, we want $\frac{1}{2}n^2 + \frac{3}{2}n \doteq 120$ secs.

$$\frac{1}{2}n^2 \doteq 100$$

$$n \doteq 14$$

$n = 14 \longrightarrow \frac{1}{2}(14^2) + \frac{3}{2}(14) = 119$ secs., 1 sec. short of 2 mins.

Thus, at 120 secs. we are just beginning the 15-second pause:
$(1+1) + (1+2) + \dots + (1+14) + 1 = 120.$

18. To determine N: Clearly, "aa...a is divisible by 7 for every a" if and only if 11...1 is divisible by 7.

Check 1 and 11 and 111, etc. for divisibility by 7; the smallest is 111,111 so $N = 6$.

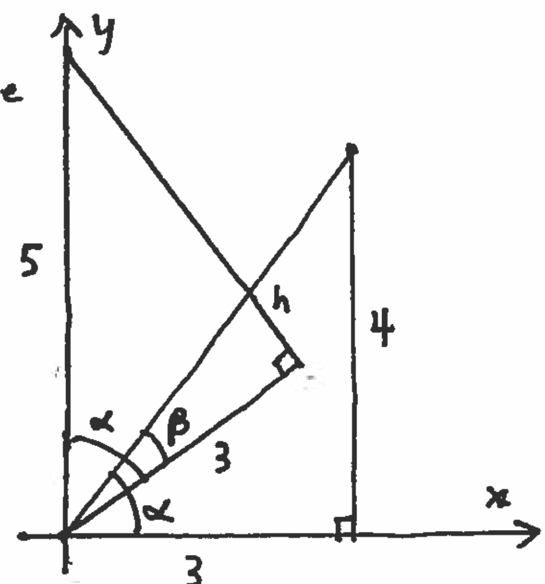
To determine M: Note $10^M = 2^M 5^M = ab$ with neither a nor b having any 0 digits implies that $a = 2^M$ and $b = 5^M$ (or vice versa).

Check to find:

- The first power of 2 with a 0 digit is $2^{10} = 1024$.
- The first power of 5 with a 0 digit is $5^8 = 390{,}625$.

Thus, $M = 8$ and $M + N = 8 + 6 = 14$.

19. Let α = angle opposite leg 4
and β = small angle in intersection triangle



Since $\tan\alpha = \frac{4}{3}$, $\alpha = \tan^{-1} 4/3$.

Since $\alpha + \alpha - \beta = 90°$ angle between x- and y-axes,

$\beta = 2\alpha - 90°$

$\beta = 2\tan^{-1} 4/3 - 90°$

area $= \frac{1}{2} bh = \frac{1}{2} \cdot 3 \cdot 3 \tan\beta$

$A = \frac{9}{2} \tan\left(2\tan^{-1} 4/3 - 90°\right)$

Method (i) Using a calculator, $A = 1.3125 = 21/16$.

Method (ii) $\beta = 2\tan^{-1} 4/3 - 90° = \tan^{-1} 4/3 + \left(\tan^{-1} 4/3 - 90°\right)$

$\beta = \tan^{-1} 4/3 - \tan^{-1} 3/4$

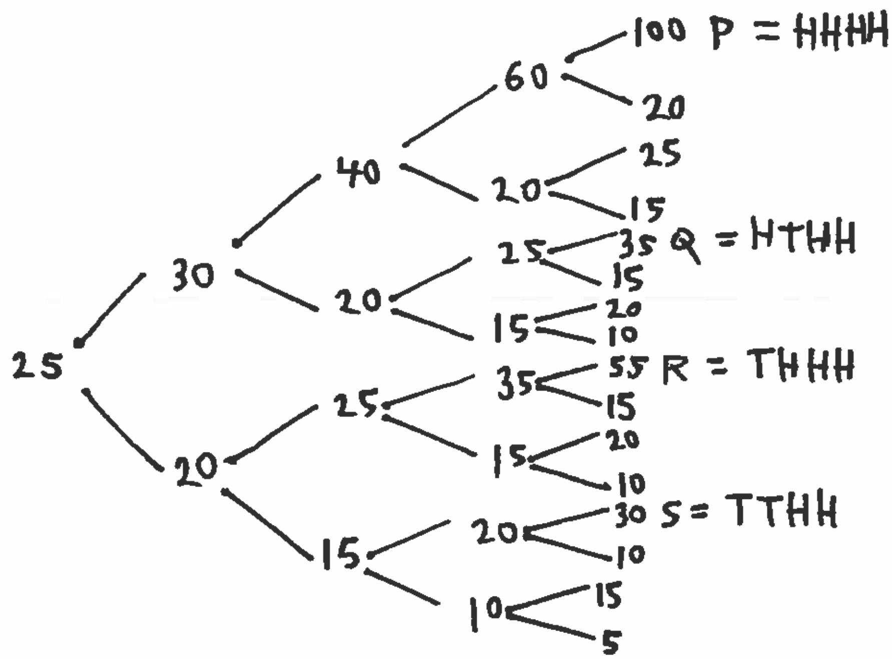
$\tan\beta = \dfrac{4/3 - 3/4}{1 + \frac{4}{3} \cdot \frac{3}{4}}$ using angle difference identity for tangent

$\tan\beta = 7/24$

$A = \frac{9}{2} \tan\beta = \frac{9}{2} \cdot \frac{7}{24} = 21/16$.

20. The first 4 flips can be sketched as a tree diagram, where each number represents her current dollars, each up-stroke is Heads, H (40%), and each down-stroke is Tails, T (60%).

After 4 flips, in only four cases (labeled P, Q, R, S) does she have enough money to reach \$100 within 3 more flips:



Investigating Q, R, S further, we find that they reach \$100 if and only if appended with 3, 2, or 3 more Heads, respectively.

Thus, $\Pr(\text{quit with} \geq \$100) = \Pr(HHHH) + \Pr(HTHHHHH)$

$$+ \Pr(THHHHH) + \Pr(TTHHHHH)$$

$$= .4^4 + .6(.4^6) + .6(.4^5) + .6^2(.4^5)$$

$$\doteq .038$$