Solutions – Fall 2017 AMATYC Student Math League Contest

1. $(-2)\Delta 5 = -2(5^2) - |-2| = -50 - 2 = -52$
$(-52)\Delta(-1) = -52(1^2) - |-52| = -52 - 52 = -104$

2. Use a "greedy algorithm": at each step, use the unit fraction that makes the cumulative total closest to 1 without reaching 1.

$\therefore \frac{1}{a} < 1$, so $\frac{1}{a} = \frac{1}{2}$

$\frac{1}{b} < 1 - \frac{1}{2} = \frac{1}{2}$, so $\frac{1}{b} = \frac{1}{3}$

$\frac{1}{c} < 1 - (\frac{1}{2} + \frac{1}{3}) = \frac{1}{6}$, so $\frac{1}{c} = \frac{1}{7}$

$n = \frac{1}{2} + \frac{1}{3} + \frac{1}{7} = \frac{41}{42}$.

3. Ways to win at least 50¢:

$\leq 2Q$	$\leq 3D$	$\leq 1N$	$\leq 4P$	prob.
2	any	any	any	$.5^2$
1	3	any	any	$.5^1 \times .5^3$
1	2	1	any	$.5^1 \times \underbrace{{}_3C_2}_{\text{no. of ways to pick 2 out of 3}} \underbrace{(.5^3) \times .5^1}_{\text{2 Heads and 1 Tail}}$

Total prob. $= .5^2 + .5^4 + 3(.5^5)$
$= 13/32$
$\doteq 0.41$

4. Including 13578, there are $4! = 24$ permutations that begin with 1.
Thus, 1 contributes 24×10^4 in those permutations,
and similarly 24×10^3 in the permutations in which 1 is the second digit; etc.
Thus, 1 contributes a total of $24(10^4 + 10^3 + \cdots + 1) = 24(11{,}111)$.
Likewise, 3 contributes a total of $3 \cdot 24(11{,}111)$; etc.
Grand total $= (1+3+5+7+8)(24)(11{,}111) = 24^2(11{,}111) = 6{,}399{,}936$

5. $M = 150 - 9 = 141$ (if 10 socks are left in the bin then they might all be pink, but if only 9 are left then that's impossible).
$m = 1 + \text{no. of colors} = 1 + 5 = 6$

$141 \times 6 = 846$.

6. $(\sin x + \cos x)^2 = \frac{1}{16}$

$\sin^2 x + \cos^2 x + 2\sin x\cos x = \frac{1}{16}$

$2\sin x\cos x = -15/16$

$\sin x\cos x = -15/32$

$(\sin x + \cos x)^3 = \frac{1}{64}$

$\sin^3 x + \cos^3 x + 3\sin x\cos x(\sin x + \cos x) = \frac{1}{64}$

$\sin^3 x + \cos^3 x + 3(-15/32)(1/4) = \frac{1}{64}$

$\therefore \sin^3 x + \cos^3 x = \frac{1}{64} + \frac{3}{4}\cdot\frac{15}{32} = \frac{47}{128}.$

Method 2 : TI-84+ calculator, \Y1 = sin(X) + cos(X) − 1/4

MODE RADIAN

Use CALC zero to find the root X ≐ 2.1785 radians

Now calculate sin (X)^3 + cos (X)^3 = .3671875

MATH ▶Frac gives 47/128.

7. Since $3^7 > 2017$, then $z = 1$ or 2.

So $x^2 + y^2 = 2017 - z^7 =$ either 2016 or 1889.

On TI-84+ calculator, \Y1 = √(2016−X²)

Examine TABLE: no integer solutions.

\Y1 = √(1889−X²) Examine TABLE: if X = 17 then Y1 = 40.

Thus $17^2 + 40^2 + 2^7 = 2017$, and we have $17 + 40 + 2 = 59$.

8. Let $n(T)$ denote the number of 4-tuples satisfying $a+b+c+d = T$.

To find, say, $n(9)$: imagine selecting any 3 tokens from a row of 12 tokens. The 3 chosen tokens partition the 9 unchosen ones into 4 subsets (some of them possibly empty) that lie to the left, right, or in-between the chosen ones. The respective sizes of the 4 subsets make a 4-tuple satisfying $T = 9$, so $n(9)$ = no. of ways to select 3 of 12 tokens $= {}_{12}C_3$.

The grand total is $n(0) + n(1) + n(2) + \cdots + n(14)$

$= {}_3C_3 + {}_4C_3 + {}_5C_3 + \cdots + {}_{17}C_3$

$= {}_{18}C_4$ by the "hockey stick identity" in Pascal's Triangle

$= 3060.$

9. Denote knight, knave, spy by T, F, M.
The three statements are,

$$\begin{aligned} X &: X \neq M \\ Y &: X = F \\ Z &: Y = M. \end{aligned}$$

Only Choice C, i.e., $(X, Y, Z) = (T, F, M)$, is consistent with all 3 statements.

10. Denote the 5 solutions by $r_1, r_2, \ldots, r_5$.
In any product $(x-r_1)(x-r_2)\cdots(x-r_5)$, the trailing coëfficient is the only 5-fold product of roots, $-r_1 r_2 r_3 r_4 r_5$.
And the coëfficient of x is the sum of all 4-fold products.

Here, $8 = -r_1 r_2 r_3 r_4 r_5$ and $-12 = r_1 r_2 r_3 r_4 + \cdots + r_2 r_3 r_4 r_5$,

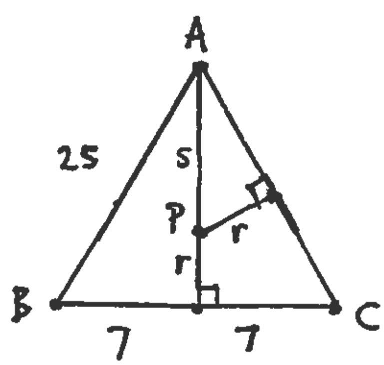
so $$\frac{1}{r_5} + \frac{1}{r_4} + \cdots + \frac{1}{r_1} = \frac{r_1 r_2 r_3 r_4 + \cdots + r_2 r_3 r_4 r_5}{r_1 r_2 r_3 r_4 r_5} = \frac{-12}{-8} = \frac{3}{2}.$$

11. Visualize the $2\times n$ grid as 2 units tall and n units wide.
Every way to cover it with dominos is formed in one of two mutually exclusive ways:

- append an extra vertical domino on the left side of any $2\times(n-1)$ covering
- append an extra pair of horizontal dominos on the left side of any $2\times(n-2)$ covering.

Thus, $T_n = T_{n-1} + T_{n-2}$ and we have $T_1 + \cdots + T_6 = 1+2+3+5+8+13 = 32$.

12. The height of ΔABC is $\sqrt{25^2 - 7^2} = 24$.
Denote the required distance by r, and the height by $r+s$.



By similar triangles, $\frac{r}{s} = \frac{7}{25}$,

$$\begin{aligned} \text{so } s &= \tfrac{25}{7} r \\ r+s &= \tfrac{32}{7} r = 24. \end{aligned}$$

$$\therefore \quad r = 7 \cdot \frac{24}{32} = \frac{21}{4}.$$

13. $$\begin{aligned} p^3 + (1-p)^3 &= \tfrac{1}{2} \\ p^3 + (1-3p+3p^2-p^3) &= \tfrac{1}{2} \\ 3p - 3p^2 &= \tfrac{1}{2} \\ 3p(1-p) &= \tfrac{1}{2} \\ p(1-p) &= \tfrac{1}{6}. \end{aligned}$$

14. Imagine listing the data from low to high: one 1 at the top, followed by two 2's, four 3's, etc.
The number of items less than 50 is $1+2+4+8+\cdots+2^{48} = 2^{49}-1$.
Below those, there is nothing except a 50 followed by $2^{49}-1$ other 50's.
Thus, the number in the middle is 50.

15. $$g(3) = g\left(\frac{1}{1/3}\right) = \frac{(1/3)^2}{2+1/3} = \frac{1}{21}.$$

$$g(g(3)) = g\left(\frac{1}{21}\right) = \frac{21^2}{2+21} = \frac{441}{23}.$$

16. All 4-digit numbers of form aaaa are multiples of $1111 = 11(101)$, so $N = 101$.

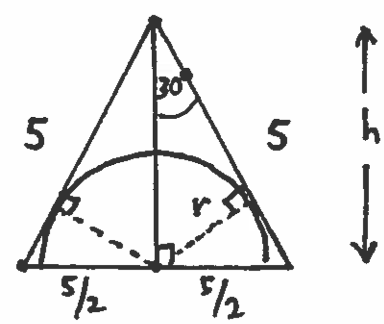
M can be found by trial and error or by the "Frobenius Number" theorem: If the greatest common divisor of a and b is 1, then the largest integer not of form $ax+by$ for any x and y is $ab-(a+b)$. Here, $M = 3\cdot 7-(3+7) = 11$.

So $N+M = 101+11 = 112$.

17. Statement I is true: If both solutions are rational then their difference is rational. But by Quadratic Formula, the difference is $\sqrt{b^2-4ac}/a$. Since "a" is rational, this forces $\sqrt{b^2-4ac}$ to be rational, so b^2-4ac must be the perfect square of a rational number.

Statement II is false: e.g., consider $x^2+\sqrt{5}x+1=0$. The solutions are both irrational, yet $b^2-4ac=1$ is the square of a rational number.

18. Denote the height of the triangle by h, and the radius of the semicircle by r.



Note $h = 5\cos 30° = 5\cdot\frac{1}{2}\sqrt{3}$,

and $r = h\sin 30° = h\left(\frac{1}{2}\right) = \frac{5}{4}\sqrt{3}$.

Area $= \frac{1}{2}\pi r^2 = \frac{1}{2}\pi\left(\frac{5}{4}\sqrt{3}\right)^2 = \frac{75}{32}\pi$.

19. Dividing x^2-16 by $ax+b$ either "longhand" or "synthetically" gives:

$$\begin{array}{c|ccc} & 1 & 0 & -16 \\ \hline -b/a & 1 & -b/a & \frac{b^2}{a^2}-16 \end{array}$$

$ax+b = a(x+b/a)$

$1/a\,(1, -b/a) = (1/a, -b/a^2)$

$$\frac{x^2-16}{ax+b} = \frac{1}{a}x + \frac{-b}{a^2} + \frac{\frac{b^2}{a^2}-16}{ax+b}$$

, so the asymptote is:

$$\frac{1}{a}x + \frac{-b}{a^2} = 3x+7$$

$\therefore \frac{1}{a} = 3$ so $a = 1/3$

$\therefore \frac{-b}{a^2} = -9b = 7$ so $b = -7/9$.

$\therefore f(x) = \dfrac{x^2-16}{\frac{1}{3}x - \frac{7}{9}}$, $f(-3) = \dfrac{9-16}{\frac{1}{3}(-3)-\frac{7}{9}} = \dfrac{63}{16}$.

20. No odd integer would be counted, since its number of even factors is 0 but its number of odd factors is positive.

No multiple of 4 would be counted, since the number of odd factors of $4n$ is the number of odd factors of n, whereas the number of even factors of $4n$ is at least twice that many, since 4 has even factors 2 and 4.

Every other even number is of form $2 \times \text{odd} = 2(2n+1)$. Every odd factor of $2n+1$, when doubled, is an even factor of $2(2n+1)$; and conversely every even factor of $2(2n+1)$, when halved, is an odd factor of $2n+1$. Thus, the numbers of odd and even factors of $2(2n+1)$ are equal.

Thus, we need only count how many numbers of form $2(2n+1) = 4n+2$ lie between $2 = 4(0)+2$
and $998 = 4(249)+2$,
so the answer is 250.