Solutions – February/March 2016 AMATYC Student Mathematics League Contest

1. $$\frac{.04d + .10(2d)}{3d} = \frac{.24d}{3d} = .08$$

2. Let g = price of gallon of gas, q = price of quart of oil.
Then
$$g = 1.2 + q = 1.2 + .6g$$
$$.4g = 1.2$$
$$g = 3$$
Thus, $2(g+q) = 2(3 + .6 \cdot 3) = \$9.60$

3. The interior angles are $180 - x$, $180 - (x+10)$, and $180 - (x-15)$.
Adding these and equating to 180 gives $x = 121\frac{2}{3}°$, inside $[120, 123)$.

4. Let n = initial number of nickels (and dimes).
Spending 15¢ reduces the number of nickels by 1 or 3, and dimes by 1 or 0, respectively. So either $2(n-1) = n-1$, leaving Ed with no coins
or $2(n-3) = n-0$, so $n = 6$ nickels.

5. $A = 1 \longrightarrow 12 = B(1^B)$, impossible for a digit B.

$A = 2 \longrightarrow 24 = B(2^B)$, possible for $B = 3$, so $A = 2$ works.

6. Let T = knight. F = knave.
The one who spoke cannot be T, since everyone else sees him, making his statement false.
Thus he's an F. Then not everyone can be F, since his statement would be true.
And there can't be only one T, since that person would see only F's, making the statement true.
Thus, there must be at least two T's.

7. Let $a, b, c, d = 17$ represent length, width, height, diameter.
Then $a^2 + b^2 + c^2 = 17^2 = 289$, with exactly two solutions.

Recall Pythagorean triples $(3, 4, 5)$ and $2(3,4,5)$ and $3(3,4,5) \cdots$
and $(8, 15, 17)$.
$(9, 12, 15)$

Thus $17^2 = 8^2 + 15^2 = 8^2 + (9^2 + 12^2)$.

cont'd

7 (cont'd)

Also notice $17^2 = 289 = 288 + 1 = 2(144) + 1$
$= 2(12^2) + 1 = 12^2 + 12^2 + 1^2$.

Thus, the two boxes are $12 \times 12 \times 1$ and $12 \times 9 \times 8$ and their volume ratio is $\dfrac{12 \times 9 \times 8}{12 \times 12 \times 1} = 6$.

Note: the two solutions can also be searched for systematically using a programmable calculator.

8. Given $527/N$ has remainder r
& $622/N$ " " "
& $698/N$ " " "
So $\dfrac{622-527}{N}$ has remainder $r - r = 0$
& $\dfrac{698-622}{N}$ " " " " .
Thus, N is a factor of both $622 - 527 = 95 = 5 \times 19$
and $698 - 622 = 76 = 2 \times 2 \times 19$,
meaning $N = 1$ or 19. But $N = 1$ would leave remainders 0, so $N = 19$.

By the same method, we find $M = 29$. Thus $M + N = 29 + 19 = 48$.

9.
$$\begin{cases} 100a + 10b + c = 29(a+b+c) \longrightarrow 71a + {}^{-}19b + {}^{-}28c = 0 \\ 100b + 10c + a = 68(a+b+c) \longrightarrow {}^{-}67a + 32b + {}^{-}58c = 0 \\ 100c + 10a + b = 14(a+b+c) \longrightarrow {}^{-}4a + {}^{-}13b + 86c = 0. \end{cases}$$

Solve this system of linear equations (e.g., calculator "rref") to get $a = 2$, $b = 6$, $c = 1$, thus $a + 2b + 3c = 2 + 12 + 3 = 17$.

10. If 3 pairs of sides are congruent, then the triangles would be congruent. Thus, only 2 pairs of sides and all 3 pairs of angles are congruent. This means the triangles are similar, differing only by a scaling factor.
Notice $18/12 = 12/8 = 3/2$,
and $\frac{3}{2}(8, 12, 18) = (12, 18, 27)$;
the triangles $(8, 12, 18)$ and $(12, 18, 27)$ are similar but not congruent.
Thus, the longest side of ΔTYC is 27.

11. Note that $ab - (a+b) = (a-1)(b-1) - 1$, so the number of ways to write $N = ab - (a+b)$ is the same as the number of ways to write $N+1 = (a-1)(b-1)$.

The smallest whole number that can be written in 3 different ways as a product of 2 whole numbers is $12 = 1 \times 12 = 2 \times 6 = 3 \times 4$, so $N+1 = 12$ and $N = 11$.

$$\left[\text{Note } \begin{aligned} 11 &= 2(13) - (2+13) \\ 11 &= 3(7) - (3+7) \\ 11 &= 4(5) - (4+5). \end{aligned}\right]$$

12. Call the quotient $Q(x)$ and remainder $R(x)$, where $\deg(R) \le 1$ so R is of the form $mx+b$.

Then $x^{2000} - 2x^{15} + 2 = (x^2-1)\,Q(x) + R(x)$.

Evaluate:

$$\left.\begin{aligned} x=1 &\longrightarrow & 1 &= R(1) \\ x=-1 &\longrightarrow & 5 &= R(-1) \end{aligned}\right\} \quad m = \frac{5-1}{-1-1} = -2$$

$\therefore R(x) = -2x + b$

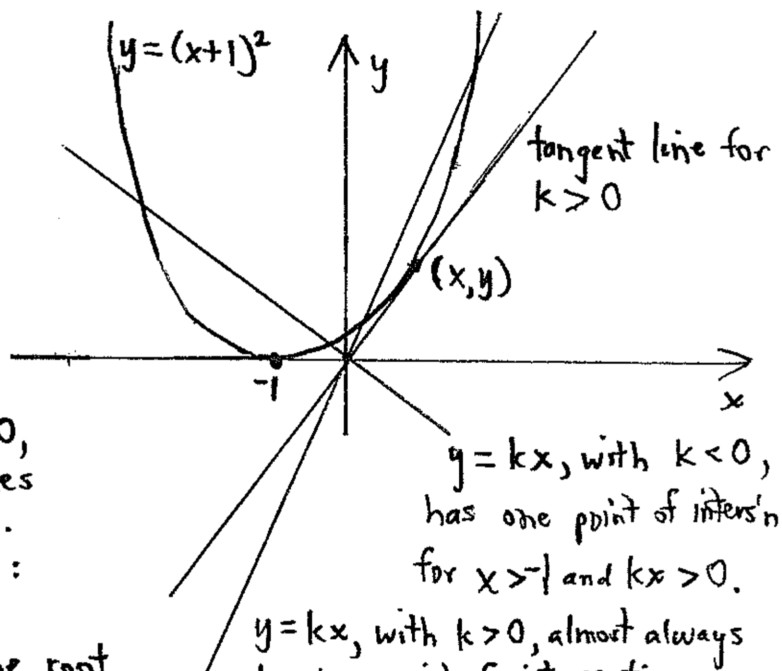
$\therefore \quad 1 = -2(1) + b \longrightarrow b = 3$

Thus, $R(x) = -2x + 3$.

13. If $\log(kx) = 2\log(x+1)$, with $x > -1$ and $kx > 0$,

then $\log(kx) = \log(x+1)^2$

$kx = (x+1)^2$.



Graphically, note that the line $y=kx$ intersects the parabola $(x+1)^2$ in exactly one point (satisfying all conditions) for every slope $k<0$, and for one slope $k>0$ that makes the line tangent to the parabola.

At the point of tangency (x,y):

$y = kx$ and $y = (x+1)^2$ has one root

$\therefore \quad kx = (x+1)^2$ " " "

$\therefore \quad x^2 + (2-k)x + 1 = 0$ " " "

$\therefore \quad$ discriminant $b^2 - 4ac = 0$

$(2-k)^2 - 4 = 0$

$2-k = \pm 2$

$k = 4$ (or $k=0$, extraneous since $kx>0$).

Thus, the answer is: $k=4$ or $k<0$.

14. Add and subtract the pair of equations and factor the results:

$$(z-x)(y-1)=2 \quad \text{and} \quad (z+x)(y+1) = 212 = 2\times 2\times 53.$$

In the first, $y-1=\pm1, \pm2 \longrightarrow y = 2, 0, 3,$ or -1.

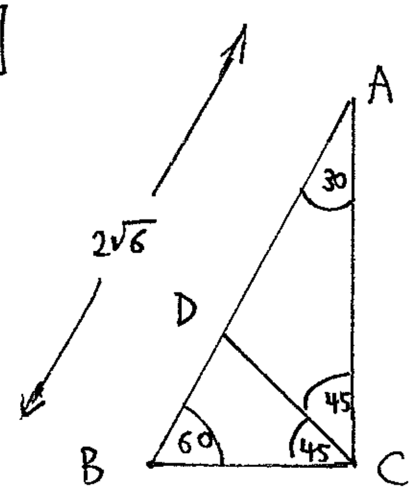
(Note that all are too small to make $y+1$ a multiple of 53.)

$\therefore$ In the second, $y+1 = \pm1, \pm2, \pm4 \longrightarrow y = 0, -2, 1, -3, 3,$ or -5.

Comparing the two lists of options for y, the only overlap is $y=0$ or 3, so there are two possible values.

[Specifically, $y=0 \longrightarrow x = 107,\ z = 105$
$y = 3 \longrightarrow x = 26,\ z = 27$.]

15. Use Law of Sines:



- in ΔACD, $\dfrac{AD}{\sin 45} = \dfrac{CD}{\sin 30}$

  so $AD = \sqrt{2}\,CD$

- in ΔBCD, $\dfrac{BD}{\sin 45} = \dfrac{CD}{\sin 60}$

  so $BD = \frac{1}{3}\sqrt{6}\,CD$.

Thus, $2\sqrt{6} = AB = AD + BD = \sqrt{2}\,CD + \frac{1}{3}\sqrt{6}\,CD = \left(\sqrt{2}+\frac{1}{3}\sqrt{6}\right)CD$

$$CD = \frac{2\sqrt{6}}{\sqrt{2}+\frac{1}{3}\sqrt{6}} = 3\sqrt{3}-3.$$

16.

$$\sin x\cos x = 4(\sin x + \cos x)$$
$$(\sin x\cos x)^2 = 16(\sin^2 x + \cos^2 x + 2\sin x\cos x)$$
$$= 16(1 + 2\sin x\cos x)$$
$$u^2 = 16(1+2u) \quad \text{where } u = \sin x\cos x$$
$$u^2 - 32u - 16 = 0$$
$$\therefore \sin 2x = 2\sin x\cos x = 2u = 2\left(\frac{32 \pm 8\sqrt{17}}{2}\right) = 32 \pm 8\sqrt{17}.$$

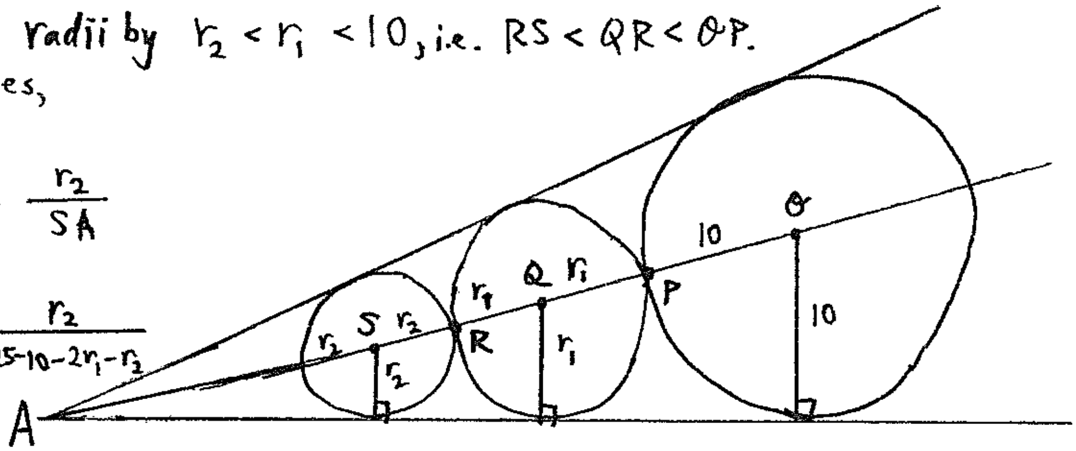
Note $32 + 8\sqrt{17} > 1$, extraneous.

Thus $a = 32$, $b = -8$, $c = 17$ and $a+b+c = 41$.

17. Denote the smaller radii by $r_2 < r_1 < 10$, i.e. $RS < QR < OP$.
By similar triangles,

$$\frac{10}{OA} = \frac{r_1}{QA} = \frac{r_2}{SA}$$

$$\frac{10}{25} = \frac{r_1}{25-10-r_1} = \frac{r_2}{25-10-2r_1-r_2}$$



But $\frac{10}{25} = \frac{r_1}{25-10-r_1} \longrightarrow .4(15-r_1) = r_1 \longrightarrow r_1 = 30/7$.

And $\frac{10}{25} = \frac{r_2}{25-10-2r_1-r_2} \longrightarrow .4(15-60/7-r_2) = r_2 \longrightarrow r_2 = 90/49$.

Thus, $AS = SA = \frac{r_2}{.4} = \frac{5}{2}(90/49) = 5(45/49)$, so $\sqrt{AS} = 15/7$.

[Shortcut: in any angle θ, such a sequence of tangent circles grows geometrically with ratio $r = \frac{1+\sin\theta}{1-\sin\theta} = \frac{1+10/25}{1-10/25} = \frac{7}{3}$.

Thus, $AS \times \frac{7}{3} \times \frac{7}{3} = 25$, so $\sqrt{AS} = 5 \times \frac{3}{7} = \frac{15}{7}$.]

18. $$\begin{aligned} a^4+b^4 &= PM^4 - QM^2N + RN^2 \\ &= P(a+b)^4 - Q(a+b)^2(ab) + R(ab)^2 \\ &= P(a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4) \\ &\quad -Q(a^3b + 2a^2b^2 + ab^3) \\ &\quad +R(a^2b^2) \end{aligned}$$

$$\therefore \begin{cases} 1 = P \\ 0 = 4P - Q \\ 0 = 6P - 2Q + R \\ 0 = 4P - Q \\ 1 = P \end{cases}$$

Solving this gives $P=1$, $Q=4$, $R=2$
so $P+Q+R = 7$.

[Shortcut on next page.]

18. [Shortcut: note $a^4 + b^4 = PM^4 - QM^2N + RN^2 = P + Q + R$, provided that $M = 1$ and $N = -1$.

But $\left\{ \begin{array}{l} a+b=1 \\ ab=-1 \end{array} \right\} \rightarrow a(1-a) = -1$

$$a^2 - a - 1 = 0$$

$$a = \frac{1 \pm \sqrt{5}}{2}, \quad b = \frac{1 \mp \sqrt{5}}{2}$$

i.e. a, b are φ and $-\varphi^{-1}$ where φ = golden ratio.

So answer is $a^4 + b^4 = \varphi^4 + (-\varphi^{-1})^4 = \varphi^4 + \varphi^{-4}$

$= 7$ using calculator

or $= F_2 + F_4 = 2 + 5 = 7$,

Since $\varphi^{2n} + \varphi^{-2n} = \underbrace{F_{2n-2} + F_{2n}}_{\text{Fibonacci numbers, } F_0=1,\ F_1=1,\ F_2=2,\text{ etc.}}$]

19. $2016 = 2^5 \cdot 3^2 \cdot 7^1 = abc$.

Think of the 3 factors as "bins" that each get any or all of the 8 prime factors; any bin that gets none is assigned a 1.

We first distribute the 2's, then the 3's, then the 7, using prime symbols (′) (one, two, or three, respectively) for these three steps.

The five 2's can be distributed to the bins in these ways (order does not matter, so we write only cases where $a \geq b \geq c$):

case	a	b	c
i	5′	0′	0′
ii	4′	1′	0′
iii	3′	1′	1′
iv	3′	2′	0′
v	2′	2′	1′

– continued –

19. (cont'd)

In case (i), the two 3's can be added to the bins in these ways:

$$\begin{array}{lll} 5'2'' & 0'0'' & 0'0'' \\ 5'0'' & 0'2'' & 0'0'' \\ 5'1'' & 0'1'' & 0'0'' \\ 5'0'' & 0'1'' & 0'1'' \end{array}$$

In the first or last of these, the one 7 can be added in 2 ways, e.g.:

$$\begin{array}{lll} 5'2''1''' & 0'0''0''' & 0'0''0''' \\ 5'2''0''' & 0'0''1''' & 0'0''0''' \end{array},$$

whereas in the others, it can be added in 3 ways, e.g.:

$$\begin{array}{lll} 5'0''1''' & 0'2''0''' & 0'0''0''' \\ 5'0''0''' & 0'2''1''' & 0'0''0''' \\ 5'0''0''' & 0'2''0''' & 0'0''1''' . \end{array}$$

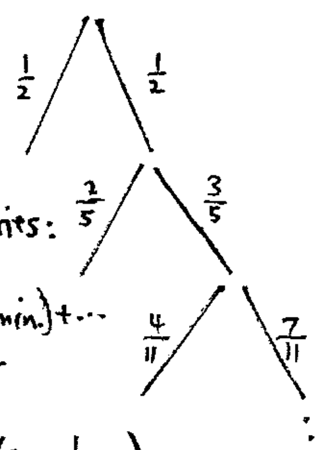
Thus, case (i) leads to a total of $2(2)+2(3)=10$ factorings.

A similar analysis shows that case (ii) leads to $3(3)+3(3)=18$ factorings.

Cases (i), (iii), (v) have the same format,
and cases (ii), (iv) have the same format.

So the total number of factorings is $3(10)+2(18)=66$.
Unfortunately, none of the answers provided is correct.

20. In the tree diagram of probabilities, each level lasts half as long as the previous level, and the left branch denotes removing the "1".



The average time is thus the sum of a convergent series:

$$T = \tfrac{1}{2}(1\text{ min.}) + \tfrac{1}{2}\cdot\tfrac{2}{5}\left(1+\tfrac{1}{2}\text{ min.}\right) + \tfrac{1}{2}\cdot\tfrac{3}{5}\cdot\tfrac{4}{11}\left(1+\tfrac{1}{2}+\tfrac{1}{4}\text{ min.}\right)+\cdots$$

$$T = \tfrac{1}{2} + \tfrac{1}{2}\cdot\left(1-\tfrac{3}{5}\right)\left(2-\tfrac{1}{2}\right) + \tfrac{1}{2}\cdot\tfrac{3}{5}\cdot\left(1-\tfrac{7}{11}\right)\left(2-\tfrac{1}{4}\right)+\cdots$$

where $a_n = \left(\frac{1}{2}\cdot\frac{3}{5}\cdot\frac{7}{11}\cdots\frac{2^{n-1}-1}{3\cdot 2^{n-2}-1}\right)\left(1-\frac{2^n-1}{3\cdot 2^{n-1}-1}\right)\left(2-\frac{1}{2^{n-1}}\right)$

Adding the terms directly or with a calculator program shows that $T \doteq 1.36$ min.
(the limit is $T \doteq 1.360486587$)