SML Round 2 2011-2012
Short Answers

1. **B** The only possible prime products are 2, 3, and 5. There are 9 possible rolls to produce these products, so $P(\text{rolling a prime number}) = \frac{9}{6^3} = \frac{1}{24}$

2. **B** $6x^3 + 5x^2 + Px + Q = (x^2+1)(6x+5)$, so $P + Q = 6 + 5 = 11$

3. **C** April 30: $04 \cdot 30$ is not a 2-digit number. February 29: $02 \cdot 29 = 58$, but **58 is not a leap year!

4. **E** $91 = (\pm 13)(\pm 7) = (a+b)(a-b)$ (91 is also $(\pm 1)(\pm 91)$, but no values for a or b would satisfy $a^2 + b^2 < 1000$). So, $(a,b) = (\pm 10, \pm 3)$ or $(\pm 3, \pm 10)$ and $a^2 + b^2 = 109$.

5. **E** For such lines, the *x*-intercept is $x = -\frac{b}{m} = \frac{m-36}{m} = 1 - \frac{36}{m}$. For this to be an integer, m must be one of the 18 integers that divide 36.

6. **B** The system $\begin{array}{r} x + \ \ y = 120 \\ 5x + 8y = 741 \end{array}$ has solution $x = 73, y = 47$. After the increases, $6(73) + 10(47) = 908$

7. **C** A little number-crunching! The largest value a can be is 12, but no integer choices of b and c work with that. When a is 11, b = 8 and c = 13 work. So $a + b + c = 32$

8. **C** Eliminating h from the system $\begin{array}{l} t + \frac{d+h}{2} = 10 \\ \frac{t+d}{2} + h = 14 \end{array}$ yields $3t + d = 6$. Then t must be 1, d=3, and h=5.

9. **A** For the Sudoku fans. Play with this until you get it!

10. **C** After some number crunching, the only solutions that work are a = 4, b =6, c = 2 and a = 5, b = 9, c = 4. 264 + 495 = 759.

11. **D** $x^2 - \frac{10}{9}x + c = (x-s)(x-s^2) = x^2 - (s^2+s)x + s^3 \Rightarrow s^2 + s = \frac{10}{9} \Rightarrow 9s^2 + 9s - 10 = 0$. The two solutions to this last equation are $s = -\frac{5}{3}$ and $s = \frac{2}{3}$. Since $c = s^3 > 0$, $s = \frac{2}{3}$ and $c = \frac{8}{27} = \frac{m}{n}$. So $m + n$ =35.

12. **C** $(x+y)^2 = x^2 + y^2 \Rightarrow 2xy = 0 \Rightarrow x = 0$ or $y = 0$ (the y and x axes)

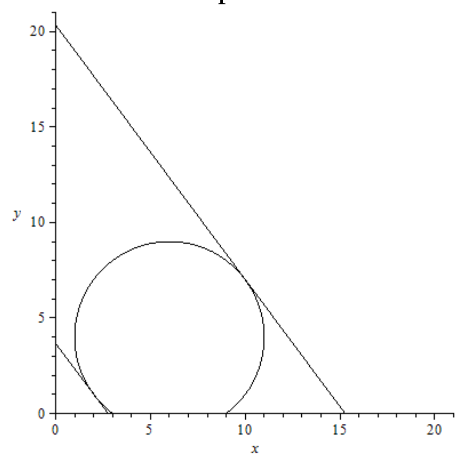
13. **D** The system $\begin{array}{l} ab + c = 29 \\ ac + b = 59 \\ ac + bc = 80 \end{array}$ yields $b(c-1) = 21$ when the 2nd equation is subtracted for the 3rd equation. Out of the 4 possibilities $(b,c) = (3,8), (7,4), (21,2), \text{and } (1,22)$, only the first one gives a solution that works: a = 7, b = 3, and c = 8.

14. Correct for all students – the correct answer of 76 was not listed. Let $C = (x, y)$. In order for the triangle to be acute, each of the following 3 inequalities must be satisfied:

$(AB)^2 + (BC)^2 > (AC)^2$
$(AB)^2 + (AC)^2 > (BC)^2$. These inequalities can be written as
$(BC)^2 + (AC)^2 > (AB)^2$

$(10-2)^2 + (7-1)^2 + (x-10)^2 + (y-7)^2 > (x-2)^2 + (y-1)^2$
$(10-2)^2 + (7-1)^2 + (x-2)^2 + (y-1)^2 > (x-10)^2 + (y-7)^2$. These give, when simplified:
$(x-10)^2 + (y-7)^2 + (x-2)^2 + (y-1)^2 > (10-2)^2 + (7-1)^2$

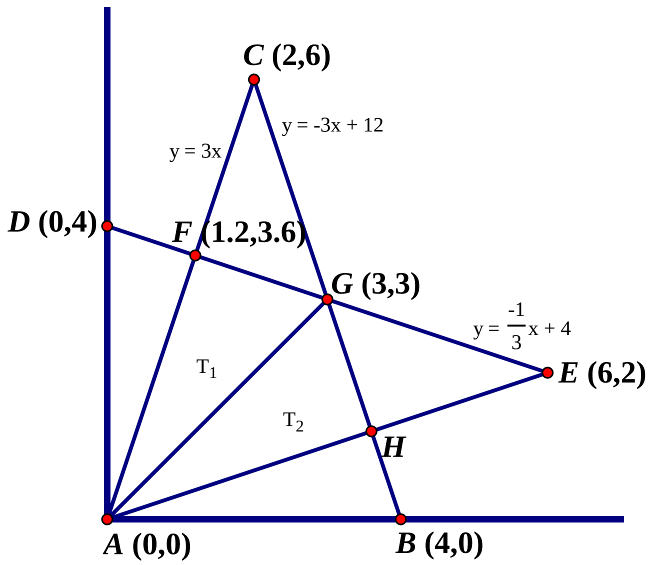
$4x+3y<61,\ 4x+3y>11$, and $(x-6)^2+(y-4)^2>25$. Graphing the system, R is the region between the lines and outside of the circular piece:



The area of this region is Area of large triangle – Area of small triangle – area of circular piece = $\frac{1}{2}\cdot\frac{61}{4}\cdot\frac{61}{3}-\frac{1}{2}\cdot\frac{11}{4}\cdot\frac{11}{3}-\int_0^9 2\sqrt{25-(y-4)^2}\,dy \approx 76$ (There are non-calculus ways of looking at this region)

15. **B** In the diagram shown, the common area is made up of triangles T_1 and T_2. Since these are congruent triangles, the area of the region common to triangles ABC and ADE is
$2\cdot T_1 = 2\cdot$(Area of triangle ADG – Area of triangle ADF)

$$= 2\cdot\left(\frac{1}{2}\cdot 4\cdot 3 - \frac{1}{2}\cdot 4\cdot\frac{6}{5}\right) = \frac{36}{5} = 7.2$$



16. **B** 1,2,3,4, and 5 all have the same representation in base 9 as they do in base 6. For 2-digit numbers, we have $ab_6 = ba_9 \Rightarrow 6a + b = 9b + a \Rightarrow 5a = 8b \Rightarrow a = \frac{8}{5}b.$ Keeping in mind that *a* and *b* must be less than 6, there are no 2-digit numbers. For 3-digit numbers, we have $abc_6 = cba_9 \Rightarrow 36a + 6b + c = 81c + 9b + a \Rightarrow 35a = 3b + 80c$. Trying $a = 1, 2, 3, 4,$ and 5, the only solution is $a = 5,\ b = 5,$ and $c = 2$ (so $552_6 = 255_9$) There are no such numbers with 4 or more digits.

17. **C** $px^2 + qx + r = (px+1)(x+r),$ or $(px+r)(x+1)$.
Case 1: $(px+1)(x+r) \Rightarrow pr + 1 = q$. This gives the triples (2,5,2), (2,7,3), (2,11,5), and so on when *p* = 2. We also have (3,7,2), (5,11,2), (11,23,2), (9,19,2) and so on for other values of *p*. 3 of these triples include 5 and 7 of these triples include 2. (One of *p* and *r* must be 2 to make *pr* + 1 = *q*)
Case 2: $(px+r)(x+1) \Rightarrow p + r = q$. Again, one of *p* and *r* must be 2. When *p* = 2, we have the triples (2,5,3), (2,7,5), (2,11,9), and so on. When *r* = 2, we have (3,5,2), (5,7,2) and so on. These add 4 more triples containing 5. So 2 and 5 appear at least 7 times.

18. **B** $\tan\left(\frac{\theta}{2}\right)=\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}=\frac{\sqrt{11}}{5}$ (because $0^\circ<\theta<90^\circ$). Let $x=\tan\left(\frac{\theta}{4}\right)$. Then

$$\frac{\sqrt{11}}{5}=\tan\left(\frac{\theta}{2}\right)=\tan\left(2\cdot\frac{\theta}{4}\right)=\frac{2\tan\left(\frac{\theta}{4}\right)}{1-\tan^2\left(\frac{\theta}{4}\right)}=\frac{2x}{1-x^2}\Rightarrow\sqrt{11}x^2+10x-\sqrt{11}=0$$

$$\Rightarrow(\sqrt{11}x-1)(x+\sqrt{11})=0\Rightarrow x=\tan\left(\frac{\theta}{4}\right)=\frac{1}{\sqrt{11}}=\frac{\sqrt{11}}{11}$$

19. **D**

	Fresh.	**Soph.**	**Jr.**	**Sr.**
Eng	1	2	1	2
CS	2	1	2	1

There are $1\cdot3\cdot3\cdot3=27$ ways to choose 3 students from 1 class level and 1 student from each of the other class levels (all majors would also be represented). Since there are 4 ways to do this, we get $4\cdot27=108$ such committees. We can also choose 2 students from 2 class levels and 1 student from each of the other class levels. There are C(4,2) = 6 ways in which to do this, but in 2 of those ways (1 Fr., 2 Soph., 1 Jr., and 2 Sr. and 2 Fr., 1 Soph., 2 Jr., and 1 Sr.), it is possible to not have each major represented. For 4 of the ways, there are $3^4=81$ such committees and for the other 2 ways, there are only 80. Total = $108+4\cdot81+2\cdot80=592$.

20. **C** The equation can be rewritten as $(y+x)(11y-x)=23$. Set up the four systems of equations $\{y+x=1,\ 11y-x=23\}$, $\{y+x=-1, 11y-x=-23\}$, $\{y+x=23, 11y-x=1\}$ and $\{y+x=-23, 11y-x=-1\}$ to get solutions (-1,2), (1,-2), (21,2), and (-21,-2).