Solutions – Winter 2018 AMATYC Student Math League Contest

1. $F_1 + F_2 = 68 \longrightarrow \left(\frac{9}{5}C_1 + 32\right) + \left(\frac{9}{5}C_2 + 32\right) = 68$

$$\frac{9}{5}(C_1 + C_2) = 4$$
$$C_1 + C_2 = 4 \cdot \frac{5}{9} = \frac{20}{9}.$$

2. Label the rings a, b, c, d from small to large. A minimum of 15 moves is needed: $(abcd, -, -) \longrightarrow (bcd, -, a) \longrightarrow (cd, b, a) \longrightarrow (cd, ab, -)$
$\longrightarrow (d, ab, c) \longrightarrow (ad, b, c) \longrightarrow (ad, -, bc) \longrightarrow (d, -, abc)$
$\longrightarrow (-, d, abc) \longrightarrow (-, ad, bc) \longrightarrow (b, ad, c) \longrightarrow (ab, d, c)$
$\longrightarrow (ab, cd, -) \longrightarrow (b, cd, a) \longrightarrow (-, bcd, a) \longrightarrow (-, abcd, -).$

3.
$$
\begin{aligned}
|4 - |3 - |2 - |1-x|||| &= 0\\
4 - |3 - |2 - |1-x||| &= 0\\
|3 - |2 - |1-x||| &= 4\\
3 - |2 - |1-x|| &= \pm 4\\
|2 - |1-x|| &= 7 \quad (-1 \text{ is extraneous})\\
2 - |1-x| &= \pm 7\\
|1-x| &= 9 \quad (-5 \text{ is extraneous})\\
1-x &= \pm 9\\
x &= 1 \pm 9 = -8 \text{ or } 10; \quad -8 + 10 = 2
\end{aligned}
$$

Another way: on TI-84 calculator,
\Y1 = abs(4 − abs(3 − abs(2 − abs(1 − X))))
GRAPH to observe two zeroes at −8, 10.

4. Dissect the pentagon into 5 isosceles triangles meeting at the center. The area of each triangle is $\frac{1}{2} \cdot \text{height} \cdot \text{base} = \frac{1}{2} \cdot h \cdot \left(2h \tan \frac{360^\circ}{10}\right)$
$$= h^2 \tan \frac{360^\circ}{10},$$
so the area of the pentagon is $5h^2 \tan \frac{360^\circ}{10}$.

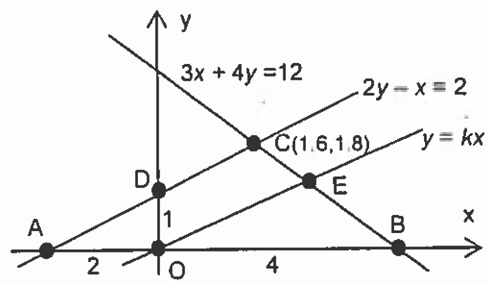
The intersection is a decagon whose 10 isosceles triangles have the same height h as before, so the decagon area is $10h^2 \tan \frac{360^\circ}{20}$.

The ratio is $10h^2 \tan \frac{360^\circ}{20} \div 5h^2 \tan \frac{360^\circ}{10} = 2 \tan 18^\circ / \tan 36^\circ$
$\doteq 89\%$ using calculator.

5. $2y - x = 2 \longrightarrow 4y - 2x = 4$

$$\begin{array}{r} -(3x + 4y = 12) \\ \hline -5x = -8 \\ x = 1.6 \\ 2y = x + 2 = 3.6 \\ y = 1.8 \end{array}$$

$\therefore C(1.6, 1.8)$



$$\begin{aligned} \text{area } BCDO &= \text{area } BCA - \text{area } ODA \\ &= \tfrac{1}{2}\cdot 6 \cdot 1.8 - \tfrac{1}{2}\cdot 2 \cdot 1 \\ &= 4.4 \end{aligned}$$

$\therefore 2.2 = \text{area } OBE$

$2.2 = \frac{1}{2}\cdot 4 \cdot \text{height}$

$1.1 = \text{height}$ so at E we have $y = 1.1$

$3x + 4(1.1) = 12$

$3x = 7.6$

$x = 7.6/3$ so $E(7.6/3, 1.1)$

So slope $k = 1.1/(7.6/3) = 33/76$.

6. $MYM^2 = AMATYC$

- Since 299^2 is only 5 digits, $M \geq 3$
- Since MYM and AMATYC have distinct units digits, $M \neq 5$ or 6.

If $M = 3$ then $M^2 = 9$ (no carry), so the hundreds digit of $AMATYC = (3Y3)^2$ must be $Y \equiv 3Y + 3Y \pmod{10}$

$\longrightarrow 0 \equiv 5Y$

$\longrightarrow Y = 2, 4, 6$ or 8

Try $Y = 2 \longrightarrow MYM^2 = 323^2 = 104329 \neq AMATYC$

$Y = 4 \longrightarrow MYM^2 = 343^2 = 117649 \neq AMATYC$

$Y = 6 \longrightarrow MYM^2 = 363^2 = 131769$ ✓

[If $M = 4, 7, 8$, or 9, it leads to a contradiction of the following form:

If $M = 4$ then $M^2 = 16$ (carry $= 1$), so the hundreds digit of $AMATYC = (4Y4)^2$ must be $Y \equiv 4Y + 4Y + 1 \pmod{10}$

$0 \equiv 7Y + 1$

$9 \equiv 7Y$,

whose only 1-digit solution is $Y = 7$, but $474^2 = 224676 \neq AMATYC$.]

7. The tangent line at $(3,4)$ is $y = mx+b$, where slope m is the negative reciprocal of $\underbrace{\text{the slope from } (3,4) \text{ to center } (-1,1)}$.

$$\frac{1-4}{-1-3} = \frac{3}{4}, \text{ so } m = \frac{-4}{3}.$$

$$y = \frac{-4}{3}x + b$$
$$4 = -\frac{4}{3}(3) + b$$
$$8 = b$$
$$m+b = -\frac{4}{3} + 8 = \frac{20}{3}.$$

8. $\{i^1, i^2, \dots, i^{2018}\} = \{i, -1, -i, 1, i, \dots, i^{2018}\}$

$\{i^{-1}, i^{-2}, \dots, i^{-2018}\} = \{\underline{-i, -1, i, 1}, -i, \dots, i^{-2018}\}$

$$0 + {}^-2 + 0 + 2 = 0$$

Every consecutive sequence of 4 terms $i^n + i^{-n}$ adds up to 0, and since $2018 = 4(504) + 2$, only the last 2 terms need be considered.

$$\begin{aligned}(i^{2017} + i^{-2017}) + (i^{2018} + i^{-2018}) &= (i^1 + i^{-1}) + (i^2 + i^{-2}) \\ &= (i + -i) + (-1 + -1) \\ &= -2.\end{aligned}$$

Another way: use a calculator with the "sum seq" feature to type: $\sum_{N=1}^{2018} (i^N + i^{-N})$.

9. Z says "I am a knave", but he can't be the knave (since knaves always lie) and he can't be the knight (since knights never lie) so Z is the spy.

X says "Z is a knight", so X cannot be the Knight (since knights never lie).
Thus, X is neither the spy nor the knight, so X is the knave,
Y is the knight,
Z is the spy.

10. Let $z = \log_A B = \log_B A$. Recall $\log_A B$ and $\log_B A$ are reciprocals,

so $z = 1/z$, thus $z = \pm 1 = \log_A B$, i.e., either $A^{+1} = B$ or $A^{-1} = B$.

But $A = B$ contradicts $A > B$,
and $A^{-1} = B$ contradicts $A > B > 1$, so there are no real solutions.

11. Since $M!(M+1)!/2 = (M!)^2(M+1)/2$ is a perfect square,
then $(M+1)/2$ must be a perfect square.
Checking downward from $M=29$: $(29+1)/2 = 15 \neq$ square,
and the largest square less than 15 is 9,
yielding $(M+1)/2 = 9 \longrightarrow M+1 = 18 \longrightarrow M = 17$.

Since $c^4 - c^2 = c^2(c+1)(c-1)$:

- it is divisible by 4, since either c is even so $c^2 = \text{even}^2$, or c is odd, so $(c+1)(c-1) = (\text{even})(\text{even})$.
- it is divisible by 3, since $\{c-1, c, c+1\}$ are three consecutive numbers, so exactly one is divisible by 3.

Thus, $c^4 - c^2$ is divisible by $4 \times 3 = 12$ for every c,
and no larger divisor works, since $c = 2 \longrightarrow c^4 - c^2 = 12$.
Thus, $N = 12$.

Thus, $M + N = 17 + 12 = 29$.

12. Let $nCr = \binom{n}{r} = \frac{n!}{r!(n-r)!}$ denote the number of combinations of r out of n items where all r items are distinct,

and $nMr = \binom{n+r-1}{n-1}$ denote the number of multisets, i.e., combinations of r out of n items where any or all of the r items may be identical.

There are 4 cases to consider, depending on the number of onion bagels selected $(3, 2, 1, 0)$.

<u>ex</u> 3 onion $\longrightarrow$ 9 other bagels, with 6 subcases:

- 5 plain and 4 others $\longrightarrow 9M4 = \binom{12}{8}$
- 4 " 5 " $\longrightarrow 9M5 = \binom{13}{8}$
- 3 " 6 " $\longrightarrow 9M6 = \binom{14}{8}$
- 2 " 7 " $\longrightarrow 9M7 = \binom{15}{8}$
- 1 " 8 " $\longrightarrow 9M8 = \binom{16}{8}$
- 0 " 9 " $\longrightarrow 9M9 = \binom{17}{8}$

By the "hockey-stick identity" in Pascal's Triangle, the sum is $\binom{18}{9} - \binom{12}{9}$.

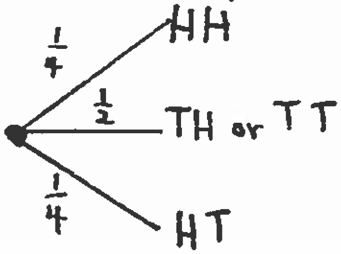
Doing this for 2, 1, or 0 onion, and summing up, we get a grand total of:

$$T = \left[\binom{18}{9} - \binom{12}{9}\right] + \left[\binom{19}{9} - \binom{13}{9}\right] + \left[\binom{20}{9} - \binom{14}{9}\right] + \left[\binom{21}{9} - \binom{15}{9}\right]$$

$$T = \left[\binom{18}{9} + \binom{19}{9} + \binom{20}{9} + \binom{21}{9}\right] - \left[\binom{12}{9} + \binom{13}{9} + \binom{14}{9} + \binom{15}{9}\right],$$ and now use hockey-stick two more times:

$$T = \left[\binom{22}{10} - \binom{18}{10}\right] - \left[\binom{16}{10} - \binom{12}{10}\right] = 594{,}946$$ with help of calculator.

13. Denote the answer by E = expected number of flips to complete first HH. Compute the expected number under 3 conditions, depending on the outcome of the first 2 flips:



HH : finished in 2 flips, so $E_1 = 2$.

TH or TT : first flip was useless, so $E_2 = 1 + E$.

HT : first 2 flips were useless, so $E_3 = 2 + E$.

$$E = p_1 E_1 + p_2 E_2 + p_3 E_3$$

$$\therefore\ E = \frac{1}{4}(2) + \frac{1}{2}(1+E) + \frac{1}{4}(2+E)$$

$$E = \frac{3}{2} + \frac{3}{4}E$$

$$\frac{1}{4}E = \frac{3}{2}$$

$$E = 6.$$

14. Denote the numbers $a < b < c < z$.
Note that $a+b < a+c$,
and $a+c < b+c$ and $a+c < a+z$;
and $b+c < b+z$ and $a+z < b+z$;
and $b+z < c+z$.
Thus:

$$\begin{cases} 2 = a+b \\ 3 = a+c \\ 4 = b+c \quad \text{or} \quad a+z \\ 5 = a+z \quad \text{or} \quad b+c, \text{ respectively} \\ x = b+z \\ y = c+z \end{cases}$$

Solving these 2 different 4×4 linear systems,

$$\left\{\begin{array}{l} 2 = a+b \\ 3 = a+c \\ 4 = b+c \\ 5 = a+z \end{array}\right\} \quad \text{and} \quad \left\{\begin{array}{l} 2 = a+b \\ 3 = a+c \\ 4 = a+z \\ 5 = b+c \end{array}\right\}$$

, for example using calculator "rref", we get $(a,b,c,z) = (\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{9}{2})$ or $(0,2,3,4)$.

So the answer is $\frac{9}{2} + 4 = \frac{17}{2}$.

15. $g(x) = \ln e^x = x$ for all real x ✓
$h(x) = e^{\ln x}$ has domain $x \geq 0$
$k(x) = \sqrt{x^2}$ has range $y \geq 0$
$m(x) = \sqrt[3]{x^3} = x$ for all real x ✓
$n(x) = 1 + (x^2-1)/(x+1)$ has domain $x \neq -1$
$p(x) = \sin(\arcsin x)$ has range $-1 \leq y \leq 1$
$q(x) = \arctan(\tan x)$ has range $-\pi/2 < y < \pi/2$
$r(x) = \pm|x|$ is not a well-defined function
$s(x) = 0.\bar{9}x = x$ for all real x ✓

3 functions are equivalent to $f(x) = x$

16. The vertex of $f(x) = x^2 - 2kx + k = (x-k)^2 + (k-k^2)$ is $(k, k-k^2)$.

The vertex of $g(k) = k - k^2 = -(\frac{1}{2} - k)^2 + \frac{1}{4}$ is $(\frac{1}{2}, \frac{1}{4})$,

So the greatest possible value of $b = k - k^2$ is $\frac{1}{4}$.

17. $abc_{10} = cba_{16}$

$$c + 10b + 100a = a + 16b + 256c$$
$$99a = 6b + 255c$$
$$33a = 2b + 85c.$$

Note $(a,b,c) = (3,7,1)$ is a solution (and no other decimal digits are solutions).
So the answer is $a + b + c = 3 + 7 + 1 = 11$.

18. $$1 + 4 + 9 + 16 + \cdots + n^2 = k^2$$
$$\frac{n(n+1)(2n+1)}{6} = k^2$$
$$\sqrt{\frac{n(n+1)(2n+1)}{6}} = k$$

On TI-84, use \Y1 = √(X(X+1)(2X+1)/6), scan TABLE for integer results. At X = 24 we see Y1 = 70, answer.

19. Consider pairs $1+99$, $2+98$, ..., $50+50$ that sum to 100.
The set can have one element from each pair, and no more, without having two numbers with a sum of 100. Thus, $M = 50$.

By the Binomial Theorem, the coëfficient of x^5 in $(x-0.5)^8$ is

$${}_8C_5 \cdot (-0.5)^3 = 56\left(\frac{-1}{8}\right) = -7. \text{ Thus, } c = -7 \text{ and } |c - M| = 57.$$

20. $$(ax^2 + bx + 4)(x+2) = x^3 + x^2 + cx + d$$

$$ax^3 + (2a+b)x^2 + \cdots = x^3 + x^2 + cx + d$$

$$\therefore \quad a = 1 \quad , \quad 2a + b = 1$$
$$b = 1 - 2a = -1$$

$$\therefore f(3) = g(3) = a(3^2) + b(3) + 4$$
$$= 1(3^2) - 1(3) + 4$$
$$= 10.$$