Appendix: Proof of the Heine–Borel Theorem

Preliminaries

First Lemma (Compact Sets are Bounded). Let $C \subseteq \mathbb{R}$ be a compact set. Then C is bounded. In other words, compact sets in $\mathbb{R}$ are bounded.

Second Lemma (Compact Sets are Closed). Let $C \subseteq \mathbb{R}$ be a compact set. Then C is closed. Hence, compact sets in $\mathbb{R}$ are closed.

Third Lemma (Closed, Bounded Sets Contain Extremes). Let $A \subseteq \mathbb{R}$ where A is nonempty. If A is closed and bounded below, then $\operatorname{glb}(A)$ exists and $\operatorname{glb}(A) \in A$. Likewise, if A is closed and bounded above, then $\operatorname{lub}(A)$ exists and $\operatorname{lub}(A) \in A$. Briefly, for any closed set, if a greatest lower bound or least upper bound exists, it must belong to that set.

Theorem

Theorem 3. (Heine–Borel). Let $A \subseteq \mathbb{R}$. A is compact if and only if A is closed and bounded.

Proof. We have already proven the easier direction of this theorem. The first and second lemmas tell us that if A is compact, then it must be both closed and bounded. The other direction of this proof gets a bit more technical.

Note, if A is empty, then A is compact ($A = \varnothing$ is certainly finite and thus compact). We will now assume that A is nonempty.

Suppose A is closed and bounded. Let $\mathcal{U}$ be an open cover of A. Since A is bounded, the greatest lower bound and least upper bound axioms guarantee that both $m = \operatorname{glb}(A)$ and $M = \operatorname{lub}(A)$ exist. Also, A is closed, so by the third lemma $m, M \in A$.

Our goal is to show that $\mathcal{U}$ has a finite subcover for A. To this end, consider the set

$$B = \{b \in \mathbb{R} \mid m \le b \text{ and } ([m, b] \cap A) \text{ is finitely covered by } \mathcal{U}\}.$$

Before moving on, let us consider exactly what the set B is. Given any $b \ge m$, $([m, b] \cap A)$ is the part of the set A which lies at or below b on the real number line. Moreover, if $b \in B$, there exists finitely many sets in $\mathcal{U}$ whose union contains $([m, b] \cap A)$. More or less, B is keeping track of how much of A can be finitely covered by $\mathcal{U}$.

Clearly, $m \in B$ since $([m, m] \cap A) = \{m\}$ is finitely covered by any single set in $\mathcal{U}$ that contains m. Thus, B is nonempty.

If this theorem is true, then *all* of A can be finitely covered by $\mathcal{U}$, so all $b \ge M$ belong to B (for such b's, we have $([m, b] \cap A) = A$). In other words, B *should* be a set that is not bounded above. For the sake of contradiction, we assume that $t = \operatorname{lub}(B)$ exists (i.e., that B is bounded above).

Case 1. This least upper bound of B is either in A or it is not, so first suppose $t \in A$. Then there exists some $V \in \mathcal{U}$ such that $t \in V$. But V is open, so there exists some open interval $(p, q) \subseteq V$ such that $p < t < q$.

Now t is the least upper bound for B. If there is no element in B between p and t, everything between p and t would be an upper bound for B, contradicting t being the *least* upper bound. Therefore, we can choose some $s \in B$ such that $p < s < t$. But $s \in B$ implies that $([m, s] \cap A)$ can be finitely covered by some $U_1, \ldots, U_N \in \mathcal{U}$. Adding V to this collection (i.e., $U_1, \ldots, U_N, V$) produces a finite subcover of $([m, r] \cap A)$ for any r with $t < r < q$. Therefore, $r \in B$ for all $t < r < q$ which contradicts t being an upper bound. Thus, t cannot belong to A.

Case 2. Suppose $t \notin A$. Then since A is closed, $(\mathbb{R} - A)$ is open, and so we can find an open interval $(p, q) \subseteq (\mathbb{R} - A)$ such that $p < t < q$. Recall that t is the least upper bound for B. If there is no element in B between p and t, everything between p and t would be an upper bound for B, contradicting t being the *least* upper bound. Therefore, we can choose some $s \in B$ such that $p < s < t$.

But again, $s \in B$ implies $([m, s] \cap A)$ can be finitely covered by some $U_1, \ldots, U_N \in \mathcal{U}$. Further, $(p, q) \subseteq (\mathbb{R} - A)$, so $([m, s] \cap A) = ([m, r] \cap A)$ for any $p < r < q$. Note that A does not contain any points in (p, q), so changing r from s within this interval has no effect on the intersection. Thus, we have that $U_1, \ldots, U_N$ also covers $([m, r] \cap A)$ for all $s < r < q$. In particular, each r, where $t < r < q$, belongs to B, since $([m, r] \cap A)$ can be finitely covered by $U_1, \ldots, U_N$. Thus t is not an upper bound for B (contradiction).

Both $t \in A$ and $t \notin A$ led to contradictions. This means t cannot exist. In other words, B cannot have a least upper bound. This means B is not bounded above.

Finally, consider $M = \operatorname{lub}(A)$. Since B is not bounded above, there is some $s \in B$ such that $s > M$. Thus $A = ([m, M] \cap A) = ([m, s] \cap A)$ is finitely covered by $\mathcal{U}$. Therefore, A is compact. ■