Chapter 3

Proficiency

Developing Students' Mathematical Knowledge

Time spent leads to experience, experience leads to proficiency, and the more proficient you are the more valuable you will be.
~Malcolm Gladwell (2008)

Students in the first two years of college are taking mathematics courses that range from basic arithmetic to differential equations. It is important that no matter the course, the primary learning outcome ought to be mathematical proficiency, the first pillar of PROWESS. Irrespective of a student's academic pursuits, mathematical proficiency is a critical component of being a productive member of society. Achieving such an outcome has a positive ripple effect. Consider Risa, a student who attained proficiency in mathematics through the help of her teacher.

Risa began her college career at a community college by taking Intermediate Algebra and, like many students, learning mathematics did not come easily for her. As a result, she was convinced that she did not have a mathematical mind since she had struggled with mathematics for as long as she could remember. During this college class the professor would lecture on the concepts and emphasize the memorization of formulas and steps to solving problems such as linear equations. Risa often asked, "When will I ever use this?", to which the instructor would respond saying that "you'll need to use this in your next math course." For Risa, this was simply insufficient for motivating her to learn mathematics. After not succeeding in Intermediate Algebra, she retook the course from another instructor, Dr. Boote, who had a very different approach to teaching students. In contrast to Risa's previous teacher, Dr. Boote focused her instruction more on making the mathematics come alive through embedding mathematical concepts within real-world contexts. For example, when learning about linear functions, Dr. Boote used the comparison of cell phone plans. "For Plan A, the initial set up charge for a cell phone plan was $25 with an additional monthly charge of $69 for unlimited data usage. For Plan B, the initial set up charge was $55 with an additional monthly charge of $59 for unlimited data usage. Which plan will be cheaper after one year? Which plan will be chapter after five years? When will the plans cost the same amount?" Dr. Boote not only embedded the mathematics within a context that was meaningful for students, but she

also probed students to justify their thinking and model the mathematics to help build a conceptual understanding. In the past, Risa had only experienced linear functions through algebraic representations, devoid of context, which added to her struggles in trying to understand the behavior of linear growth. Risa learned the "how, why, and where" a concept would be used which helped her to build a deeper understanding of the mathematics, even if it was not an area that she was going to study in the future. She used this idea of connecting classroom topics to applications in future courses. She has since earned a master's degree in sociology and teaches at a two-year college. With Dr. Boote's guidance, Risa was able to become mathematically proficient and succeeded in reaching her goals.

Characterizing Mathematical Proficiency

Dr. Boote realized that for Risa to be mathematically proficient she needed to do more than just learn mathematical procedures and that proficiency in mathematics is multifaceted. However, what does it mean to be mathematically proficient? In AMATYC's Crossroads in Mathematics, the Standards for Intellectual Development addresses desired modes of student thinking and presents goals for student learning outcomes. They identified seven specific areas of focus:

- problem solving
- modeling
- reasoning
- connecting with other disciplines
- communicating
- using technology
- and developing mathematical power (AMATYC, 1995).

Through the lens of the Standards, proficiency is characterized more broadly by not only adherence to these areas, but additionally by correctly navigating a mathematical procedure and perceiving mathematics as an enriching and empowering discipline.

In its 2001 book, *Adding It Up: Helping Children Learn Mathematics*, the National Research Council (NRC) defines mathematical proficiency as a multifaceted model with five interdependent strands essential to the learning of mathematics

- Conceptual understanding: comprehension of mathematical concepts, operations, and relations.
- Procedural fluency: skill in carrying out procedures flexibly, accurately, efficiently, and appropriately.
- Strategic competence: the ability to formulate, represent, and solve mathematical problems.
- Adaptive reasoning: the capacity for logical thought, reflection, explanation, and justification.
- Productive disposition: the habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy (p. 11).

These five areas were key components to the development of the Common Core State Standards for Mathematics (CCSSM) and the Standards for Mathematical Practice (SMPs), a variant of mathematical proficiency (National Governors Association Center for Best Practices & Council of Chief State School Officers [NGA Center & CCSSO], 2010). The SMPs are endorsed by the National Council of Teachers of Mathematics (NCTM) and state that students should be able to

- Make sense of problems and persevere in solving them.
- Reason abstractly and quantitatively.
- Construct viable arguments and critique the understanding of others.
- Model with mathematics.
- Use appropriate tools strategically.
- Attend to precision.
- Look for and make use of structure.
- Look for and express regularity in repeated reasoning (NGA Center & CCSSO, 2010, p. 6-8).

Though not all states have adopted CCSSM, a number have similar standards and outcomes. Together, the AMATYC Standards for Intellectual Development, NRC's strands for mathematical proficiency, and the mathematical practices from CCSSM, paint a picture of mathematical proficiency that goes beyond developing procedural fluency. In *IMPACT* we develop a new definition unique to the teaching and learning of mathematics during the first two years of college. CCSSM highlights *focus, coherence, and rigor* as essential principles for effective mathematics teaching and learning. *Focus* intends that faculty teach fewer topics than they have traditionally taught, *coherence* emphasizes building knowledge and understanding of mathematics within and across grades, and *rigor* aims to balance procedural fluency, conceptual understanding, and applications of mathematics (NGA Center & CCSSO, 2010). To expand on the notion of rigor, NCTM (2014) provides a definition of procedural fluency in their position statement

> Procedural fluency is a critical component of mathematical proficiency. Procedural fluency is the ability to apply procedures accurately, efficiently, and flexibly; to transfer procedures to different problems and contexts; to build or modify procedures from other procedures; and to recognize when one strategy or procedure is more appropriate to apply than another. To develop procedural fluency, students need experience in integrating concepts and procedures and building on familiar procedures as they create their own informal strategies and procedures. Students need opportunities to justify both informal strategies and commonly used procedures mathematically, to support and justify their choices of appropriate procedures, and to strengthen their understanding and skill through distributed practice (para. 1).

Proficiency in Mathematics: An Example of Number Sense and Fraction Operations

One of the missions of the two-year college is to provide pre-college level content courses, such as developmental mathematics, for students as they remediate on critical topics necessary for college-level mathematics. Students often enter the community college with deficiencies in their understanding of fraction operations, as well as number sense. Thus, it is an important goal of two-year colleges to help students build their mathematical proficiency with fractions. Consider the following example that illustrates the mathematical ideas that can be leveraged to facilitate students' development of proficiency with fractions.

When thinking about understanding fractions, students are encouraged to make sense of fractions and fraction operations from a multiplicative perspective. That is, in addition to a part-whole model, a ratio model, and a division model, the idea of fraction is developed using partitioning and iterating. For example, 11/3 is 11 copies of one-third, where 1/3 is the amount we get by taking a whole, cutting it up into 3 equal parts and taking 1 of those parts. An outline of the learning trajectory follows.

- develop a multiplicative view of fraction
- compare fractions
- create equivalent fractions
- focus on what represents one whole unit
- addition of fractions
- subtraction of fractions
- multiplication of fractions
- division of fractions including the common denominator algorithm and the invert and multiply algorithm.

Throughout all activities focused on fractions, students are encouraged to use the meaning of fraction to solve problems in ways that make sense to them. That is, the focus is not on teaching different algorithms directly but to allow students to make sense of problems, use the meaning of fraction, and to have the mathematically sound algorithms and procedural operations to emerge from their sense of meaning of fractions.

Relative to multiplication of fractions, it is common for students to learn to multiply fractions simply by "just multiplying the numerators and the denominators." For example, $\frac{7}{8} \cdot \frac{3}{4} = \frac{7 \cdot 3}{8 \cdot 4} = \frac{21}{32}$. This style of teaching leads to misconceptions:

- Students learn to view a fraction as consisting of two whole numbers rather than learning to develop a multiplicative notion of a fraction where a fraction is perceived as a single entity.
- Students fail to develop number sense. Often, students are unable to answer the following: Why does seven-eighths times three-fourths produce a result that is greater than one-half but less than one?
- Students often try to use the same multiplication procedure when adding fractions. That is, they incorrectly add the numerators, then add the denominators when adding two fractions.

When devoid of meaning and understanding, in the mind of a student, the multiplication of two fractions algorithm becomes just another of a long list of procedures to follow–procedures that may or may not make sense to the student.

Modeling with mathematics provides a means by which students can make sense of multiplication of fractions, and contribute to their development of mathematical proficiency. That is, as students engage in the modeling process, they develop meanings and understandings from which the desired procedure may emerge in a way that makes sense to them. Consider the following task: *Terry noticed that there was $\frac{3}{4}$ of a rectangular cake left after the party and that $\frac{1}{8}$ of the remaining cake had all of the frosting taken from it. What part of the original cake remaining still has frosting?* Students may be encouraged to model this situation by representing the fact that $\frac{3}{4}$ of a rectangular cake is left after the party (see Figure 1). Note that the rectangle has been cut up into 4 equal parts. Each part represents $\frac{1}{4}$ of the whole rectangle. Three of these parts are shaded.

Figure 1: Representation of ¾ of a cake.

That is, 3 copies of $\frac{1}{4}$ are shaded. We say 3 copies of $\frac{1}{4}$, or 3 one-fourths, or just $\frac{3}{4}$.

Now, $\frac{1}{8}$ of *this* amount had all the frosting removed from it. To represent this idea, we can cut the remaining cake into 8 equal parts as shown in Figure 2. One-eighth of the remaining three-fourths of the cake is represented by the top row shaded darker. Using this model, students can see the portion of the remaining cake (shaded lighter in this model) that still has frosting. They can determine that there are $7 \cdot 3 = 21$ such pieces. When compared to the entire cake (remember that we originally started with $\frac{3}{4}$ of a whole cake), the portion that still has frosting is $\frac{21}{32}$ of the entire cake. Using this method of modeling situations, students may come to understand multiplication of fractions. That is, $\frac{7}{8} \cdot \frac{3}{4}$. With repeated reasoning (SMP #8), students may come to realize that the product of the numerators $(7 \cdot 3 = 21)$ represents the number of pieces of cake under consideration in the problem (in this case, the amount of cake that still has frosting). The product of the denominator $(8 \cdot 4 = 32)$ represents the total number of pieces of cake. This leads to the traditional algorithm for multiplying fractions: $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$. With careful instruction, students can make sense of the product of two fractions by thinking about taking, for example, seven-eighths of a copy of three-fourths. When students are able to do this, they have developed a powerful way of thinking about fractions multiplicatively and increased their mathematical proficiency. Students then have the opportunity to develop a mental model for multiplication of fractions. That is, they may learn to imagine what seven-eighths of a copy of three-fourths might look like. When students are afforded the opportunity to model with mathematics, the traditional algorithms can emerge from student thinking. The algorithms then are part of a well-connected network of understanding of ideas and mental images. When this is the case, students are able to develop procedural fluency.

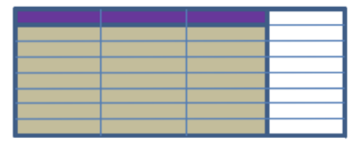


Figure 2: Representation of the cake cut into 8 equal parts.

According to AMATYC's (2006) *Beyond Crossroads*, literacy focuses on the ability to collect, organize, interpret, model, represent, and use data to help solve real-world problems. In order to develop a conceptual understanding, students must reach a level of mathematical literacy, which is defined as the capacity to identify, understand, and engage in mathematics to make well informed judgments about the role that mathematics plays (Niss, 2003). Research in mathematics education identifies that solving real-world problems is a complex process that often cannot be done quickly (Carlson & Bloom, 2005), but when students have the ability to apply mathematics to real-world problems they have moved beyond observing and executing a series of isolated skills to the realm of critical thinking.

Many collegiate programs have critical thinking as a learning outcome, and mathematics courses are key in developing such skills. AMATYC's Standards for Intellectual Development is important to building this outcome. A learning environment that promotes and cultivates critical thinking integrates learning activities and instructional strategies that reflect knowledge of students' skills, interests, cultural backgrounds, language proficiency, and individual needs. Thus, students should be encouraged to participate in learning mathematics processes that are facilitated through team-building skills, collaborative projects, portfolios, research, or field investigations. It is through solving meaningful projects and investigations that students develop the ability to think critically and apply mathematics to real-world problems.

According to Devlin (1997) and Boaler (2016), students' views on the meaning of mathematics is generally different from those of experts in the field. Mathematicians generally define mathematics as a discipline of patterns that can be considered creative and artistic, yet students often describe mathematics as a set of procedures and calculations. These two differing viewpoints underscore the need for faculty to assist students in discovering the patterns, beauty, and mathematical structure for students to move beyond the view that mathematics is simply procedures to be followed. According to the CCSSM and AMATYC's Standards for Intellectual Development, to be mathematically proficient it is necessary for students to

- know mathematics procedures and execute core computations fluently
- view mathematics as relevant to their daily lives
- demonstrate evidence of mathematical understanding
- utilize the structure in the mathematics
- make sense of and solve problems
- apply mathematics to everyday situations
- communicate mathematically and do so with precision
- defend their work and critique the work of others.

These attributes are interrelated and contribute to the development of mathematical proficiency. In the following section, we discuss how faculty can foster and assess these practices and offer suggestions for mathematics departments to assist faculty in building mathematical proficiency in students.

Fostering Mathematical Proficiency in Students

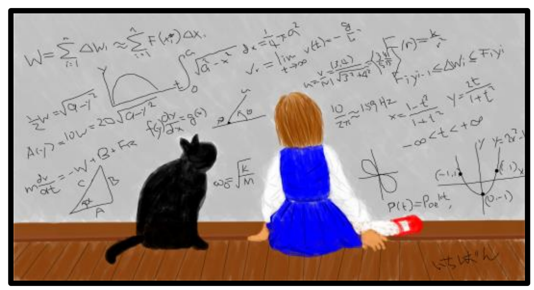
Faculty, departments, and institutions can contribute to the development of mathematical proficiency in students by creating high-quality curricula and learning environments that challenge students to think creatively and critically in and outside the classroom. Developing mathematics curricula that engender proficiency is a long-term process that leverages evidence-based research. Faculty who develop such curricula should also engage with professional organizations and collaborate with local and regional stakeholders. Suggestions are provided for creating an effective mathematics curriculum

- As a first step in curriculum design, instructors should determine the learning outcomes students need to meet at the end of a course (Wiggins & McTighe, 2005), and also consider the skills that are necessary for subsequent courses.
- The curricula and teaching processes must create and sustain equitable access to quality teaching and learning.
- Curricula sequences should be designed to support the needs of students in a wide variety of college programs and have multiple paths for different majors or student interests.
- Create curricula and activities that will foster the use of multiple approaches or representations to examine mathematical concepts.
- Curricula should promote quantitative thinking through activities that emphasize the recognition of patterns, relations, and functions.
- Students should be provided opportunities to represent and communicate mathematical ideas using multiple representations such as numerical, graphical, symbolic, and verbal.
- Curricula should provide opportunities for students to develop or improve these skills through substantive and real-life applications.

Curricula cannot be effective without fostering a productive learning environment which promotes mathematical proficiency. The development of an effective learning environment, in and outside the classroom, should be a collaborative initiative between mathematics faculty, support staff, and students. When constructing such learning environments, consider

- The learning environment must be welcoming and promote principles of inclusion, access, and equity (Chao, Murray, & Gutiérrez, 2014).
- Faculty should be mindful of how their biases and prejudices may affect student learning.
- Faculty should find a way to spur an appreciation of how mathematics can be used outside of the classroom environment (NRC, 2001).
- Faculty should be involved in training tutors specifically on how to assist students effectively in the first two years of college mathematics.
- Learning resources such as the library, disability services, student support networks, and tutoring centers should be available to students who may be in need.

It is undeniable that a supportive and caring department and faculty can be instrumental to helping a student attain mathematical proficiency. A case in point:



Anne was a single mom attending school to become an elementary school teacher. She was taking classes at the local university and needed to pass a standardized exam to become a certified teacher. Even with the help of university tutors, she was unsuccessful in passing the mathematics portion of the exam. So, she approached Kate, a soon-to-be mathematics teacher at a nearby community college, for help. Kate suggested to Anne that she focus on understanding the mathematical concepts and suggested she speak with Kate's students enrolled in the mathematics teachers' prep course. Anne found the conversation with future mathematics teachers extremely insightful and valuable and wished she had been taught such things during her educational path. Kate, with the consent of the mathematics department, invited Anne to periodically visit teacher prep courses throughout the semester to help her gain such insights and patch up the holes in her mathematics. It was here where Anne learned the building blocks of mathematics, how to be persistent in mastering the basic skills of mathematics, and how these fundamental concepts build upon one another. Kate continued to check in with Anne and invite her to class when discussing mathematical concepts that Kate knew were areas in which Anne struggled. Through these opportunities and hard work Anne eventually became mathematically proficient and passed the mathematics portion of the standardized exam. She is now a first grade teacher, and has earned her Masters in Education. It is evident that Kate's caring, a supportive department, and hard work was instrumental in Anne's acquisition of mathematical proficiency.

Creating robust curricula, interactive learning environments with an emphasis on applying mathematics, and strong student support networks will foster mathematical proficiency in students.

Examining Mathematical Proficiency

After examining how to foster mathematical proficiency in students, it is important to identify how these attributes can be assessed. *Assessment procedures* that are ongoing and provide data on the nature and quality of learning lead to improvements in student learning. This process provides the necessary data for making informed decisions about curriculum, learning environments, teaching, and program development (AMATYC, 2006). Assessment instruments and practices must be aligned with the appropriate curriculum and instruction. They should measure (1) whether students have a coherent understanding of mathematical concepts, (2) students' fluency with computational and procedural skills, (3) students' ability to utilize various problem-solving strategies, and (4) students' ability to communicate mathematical constructs numerically, analytically, graphically, and verbally.

Research points to the importance of utilizing *authentic* assessments (Silva, 2009). Authentic assessments strive to evaluate students' abilities in real-world contexts and involve multiple indicators that are relevant, meaningful, and realistic (Romberg, 1995). Their focus is on measuring analytical, collaborative, and communicative skills, as well as students' ability to utilize what they have learned in real-world situations. Classroom assessment techniques should not only be formative in nature, but diagnostic and summative as well (Boaler, 2016). Faculty and departments should use results to revise curricula and improve teaching.

Assessment results can be used to motivate faculty to review and revise the curriculum to ensure that students are mathematical proficient. The Mathematical Association of America (MAA) Common Vision document (Saxe & Braddy, 2015) provides a framework for improving the mathematical curriculum in the first two years. This framework states that instructors should intentionally plan curricula to

- Enhance students' perceptions of the beauty, vitality, and power of the mathematical sciences.
- Enhance students' understanding of mathematics as a creative endeavor.
- Increase students' quantitative and logical reasoning abilities needed for informed citizenship and for the workplace;
- Increase students' confidence and joy in doing mathematics and statistics.
- Improve students' ability to communicate quantitative ideas orally and in writing (and since a precursor to communication is understanding, improve students' ability to interpret information, organize material, and reflect on results).
- Encourage students to continue taking courses in the mathematical sciences (Saxe & Braddy, 2015, pp. 12-13).

Assessing mathematical proficiency should reveal if students are able to apply procedures appropriately, are able to demonstrate understanding of mathematics, can make sense of problems, can apply and communicate mathematics, and can justify their thinking. Meeting these criteria demonstrate that students are reaching mathematical proficiency.

Working towards Mathematical Proficiency

The unique characteristics of teaching mathematics in the first two years of college allows us the opportunity to foster mathematical proficiency by creating curricula and learning environments that meet the needs of all students while making it relevant to their everyday lives. Through the broad range of courses that we teach, from developmental mathematics to general mathematical skills to specific technical skills to advanced mathematics, we have the opportunity to make a wider impact on producing mathematically proficient citizens. Making the mathematics relevant and applying it to everyday situations is a key component in helping the students make sense of the concepts and lead to greater mathematically understanding and mathematical proficiency.

Mathematical proficiency is achieved when students know mathematical procedures and are able to execute core computations fluently and precisely, can utilize the structure of mathematics to make sense of and solve problems, are able to apply mathematics to everyday situations and defend their process, and can view concepts as relevant to their lives and communicate mathematically. This must be a collaborative effort that involves various stakeholders—faculty, departments, and support services—all focused on addressing the need of all students.

Is your interest piqued about fostering mathematical proficiency in your students? Do you already have great information or activities involving mathematical proficiency? Head to AMATYC.org/IMPACTLive and find innovations your colleagues are using or contribute innovations and ideas of your own.

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