Guided Reinvention of Sequence Convergence

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A quick survey of our audience

Have you taught any course in which you have introduced formal limit definitions?

a) Yes

b) No

Guided Reinvention

The goal is to

"design instructional activities that

a) link up with the informal situated knowledge of the students, and
b) enable them to develop more sophisticated, abstract, formal knowledge, while
c) complying with the basic principle of intellectual autonomy"

(Gravemeijer, 1998, p. 279)

Research Questions

What are the ***cognitive challenges*** that students encounter during guided reinvention of definitions of convergence?

How do the ***resolutions*** of these challenges support the students' reinvention?

Some Details

- 6-day Teaching Experiment
- 2 second-semester calculus students
- Each session was 90-120 minutes in length

Sequence Convergence

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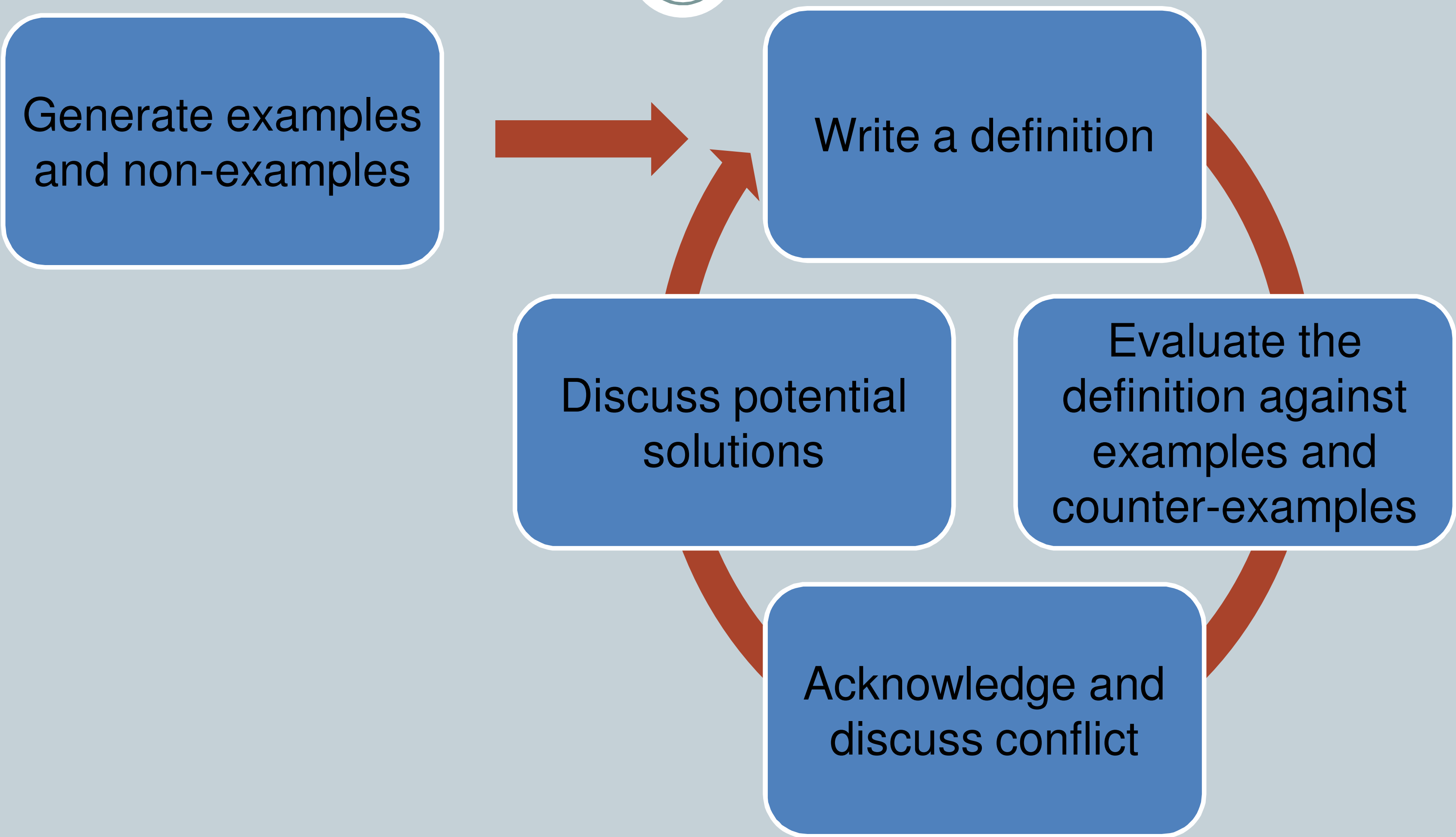
Series Convergence

↓

Point-wise convergence

Iterative Refinement



"A sequence converges to 5 as $n \to \infty$ provided…"

Definition 1: "A sequence converges to 5 as $n \to \infty$ provided $n+1$ is closer to 5 than n for any n value."

Definition 2: "A sequence converges to 5 as $n \to \infty$ provided that the number approaches or is 5 and no other number."

Definition 3: "As $n \to \infty$ the sequence approaches or becomes 5 and does not approach [stops writing]"

Definition 4: "As $n \to \infty$, at some n, a_{n+1} becomes closer to (or is) 5."

Definition 5: "As $n \to \infty$, at some n, a_{n+1} becomes closer to (or is) 5 such that: $\lim_{n \to \infty} |a_n - 5| = 0$

Definition 6: "As $n \to \infty$ the terms of the sequence become infinitely closer to 5, or actually becomes 5, where at some point n each subsequent term is closer to 5 than the previous term."

Definition 7: "As $n \to \infty$ the terms of the sequence get negligibly closer to 5 and no other number or actually become 5."

Definition 8: "As $n \to \infty$ the terms of the sequence produce an error such that $|5-a_n| \leq .01$."

Definition 9: "The terms of the sequence at some point n have an error such that $|5-a_n| \leq .01$ and never exceed that error as $n \to \infty$ after the initial n which causes the series to reach $|5-a_n| \leq .01$."

Definition 10: A sequence converges to 5 as $n \to \infty$, iff there exists n $|5-a_n| \leq .01$."

Definition 11: "A sequence converges to 5 as $n \to \infty$ i.f.f. $\exists n \in \forall$ $|5-a_n| \leq .01$."

Definition 12: "A sequence converges to 5 as $n \to \infty$ i.f.f. $\exists$ n such that for all a_n $|5-a_n| \leq .01$."

Definition 13: "A sequence converges to 5 as $n \to \infty$ i.f.f. , $\exists n_x$ such that for all a_n terms after $|5-a_n| \leq .01$."

Definition 14: "A sequence converges to 5 as $n \to \infty$ only if there exists a point n after which all a_n terms are within .01 of 5."

Definition 15: "A sequence converges to 5 as $n \to \infty$ provided that $|5-a_{n+1}| < |5-a_n|$ after some point n, and it does not converge to any other number."

Definition 16: "A sequence converges when the sequence at some number n is within $|5-a_n| < .01$ and does not exceed $|5-a_n| < .01$ for all n as $n \to \infty$."

Definition 17: "A sequence converges to 5 as $n \to \infty$ provided that $|5-a_{n+1}| < |5-a_n|$ after some point n."

Definition 18: "A sequence converges to 5 as $n \to \infty$ provided that the error does not exceed $|5-a_n| \leq .001$ and $|5-a_{n+1}| < |5-a_n|$."

Definition 19: "A sequence converges to 5 as $n \to \infty$ only if there exists a point n after which the error does not exceed $|5-a_n| \leq .001$ and $|5-a_{n+1}| < |5-a_n|$."

Definition 20: "A sequence converges to 5 as $n \to \infty$ only if there exists a value n after which the error does not exceed your chosen error and $|5-a_{n+1}| < |5-a_n|$."

Definition 21: "A sequence converges to 5 as $n \to \infty$ only if there exists a value N after which $|5-a_n|$ does not exceed every error after N and $|5-a_{n+1}| < |5-a_n|$."

Definition 22: "A sequence converges to 5 if $|5-a_n|$ does not exceed every error staring at some value N (where the sequence is within the error)."

Final Definition: "A sequence converges to U when $\forall \varepsilon > 0$ there exists some N $\forall n \geq N$ $|U-a_n| \leq \varepsilon$."

Results: Opportunity Types

“Problems” – Issues raised *by the students*, most commonly as conflicts between their concept image and their stated definition

“Problematic Issues” – Issues that arose *unbeknownst to the students*, most commonly as conflicts between **our** concept image and the students’ stated definition or concept image

Note: A “Problematic Issue” can become a “Problem”

Results: Resolution Types

Explicit: Issue is resolved via conversation focused on resolving the stated issue

Implicit: Issue is resolved “in the background” while the focus of conversation is on resolving another (related or unrelated) issue

Results: What Role Do We Play?

Guide the Process: Steer the ship toward productive waters

Produce Conflict: Ask questions designed to turn "Problematic Issues" into "Problems"

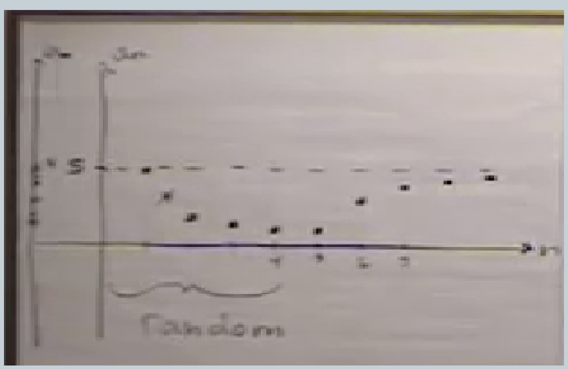
Provide Solution: At "appropriate" times, offer solutions to "Problems"

Elicit Reasoning: Ask for elaboration to produce rich data

Bad Early Behavior

Definition 2: A sequence converges to 5 as $n \to \infty$ provided that the number approaches or is 5 and no other number.

Belinda wonders aloud about Graph A



Poll

Is the issue of “Bad early behavior” a

a) Problem
b) Problematic issue
c) Explicit solution
d) Implicit solution

Problems and Solutions

Definition 2: A sequence converges to 5 as $n \to \infty$ provided that the number approaches or is 5 and no other number.

Problem: Bad Early Behavior → Explicit → Solution: "At some point *n*"

Problem: Bad Early Behavior → Explicit → Solution: "At some point *n*"

Megan: "What if we were to say **after some point *n***?...We have random stuff going on [*pointing at all of the convergent graphs*], but at some point it does start doing the closer and closer and closer thing."

Problem: Bad Early Behavior → Explicit → Solution: "At some point n"

Definition 3: As $n \to \infty$, at some n, a_{n+1} becomes closer to (or is) 5.

What problematic issues arise with Definition 3?

(Please enter your response in the chat window.)

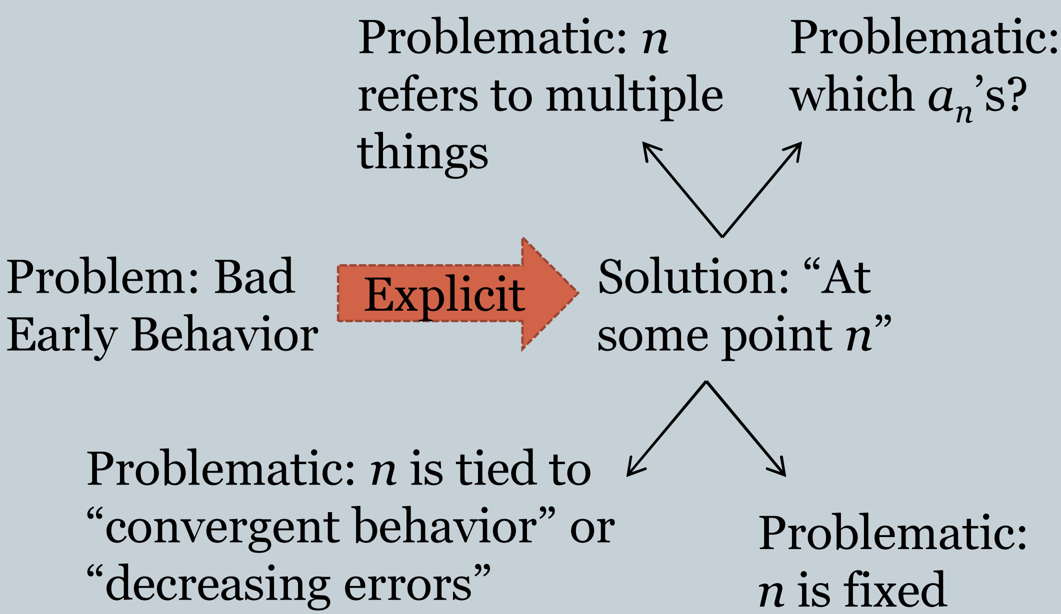
Problem: Bad Early Behavior → Explicit → Solution: "At some point n"

Definition 3: As $n \to \infty$, at some n, a_{n+1} becomes closer to (or is) 5.

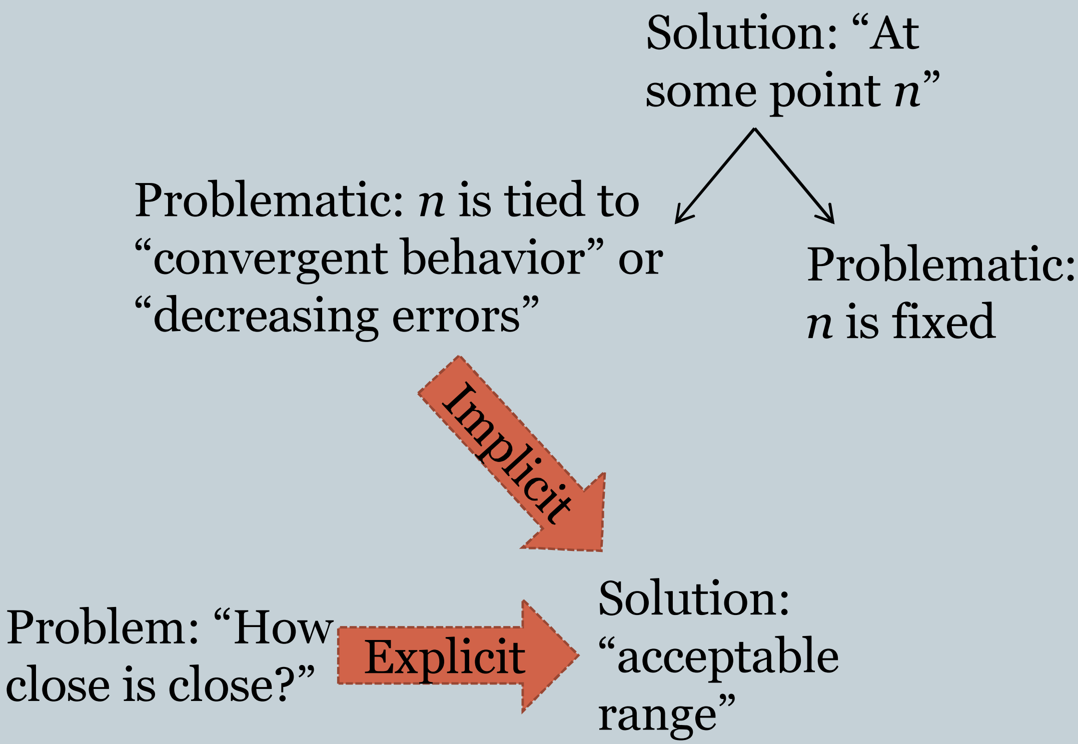
Resolutions lead to new Opportunities…

- Problematic Issue 1: n refers both to the terms of the sequence and to a fixed place on the n-axis
- Problematic Issue 2: How should we refer to the terms of the sequence after the "some point n"? a_{n+1}?
- Problematic Issue 3: The "some point n" is tied to the sequence becoming monotonic.
- Problematic Issue 4: The "some point n" is fixed for any given sequence.

Problems and Solutions



Problems and Solutions



Problem: "How close is close?" → Explicit → Solution: "acceptable range"

Belinda: An error exists, but it's...

Megan: Um... We were doing that stuff before where our approximation had an error. Remember when we had to find it within .0001, or whatever?

Belinda: Yeah.

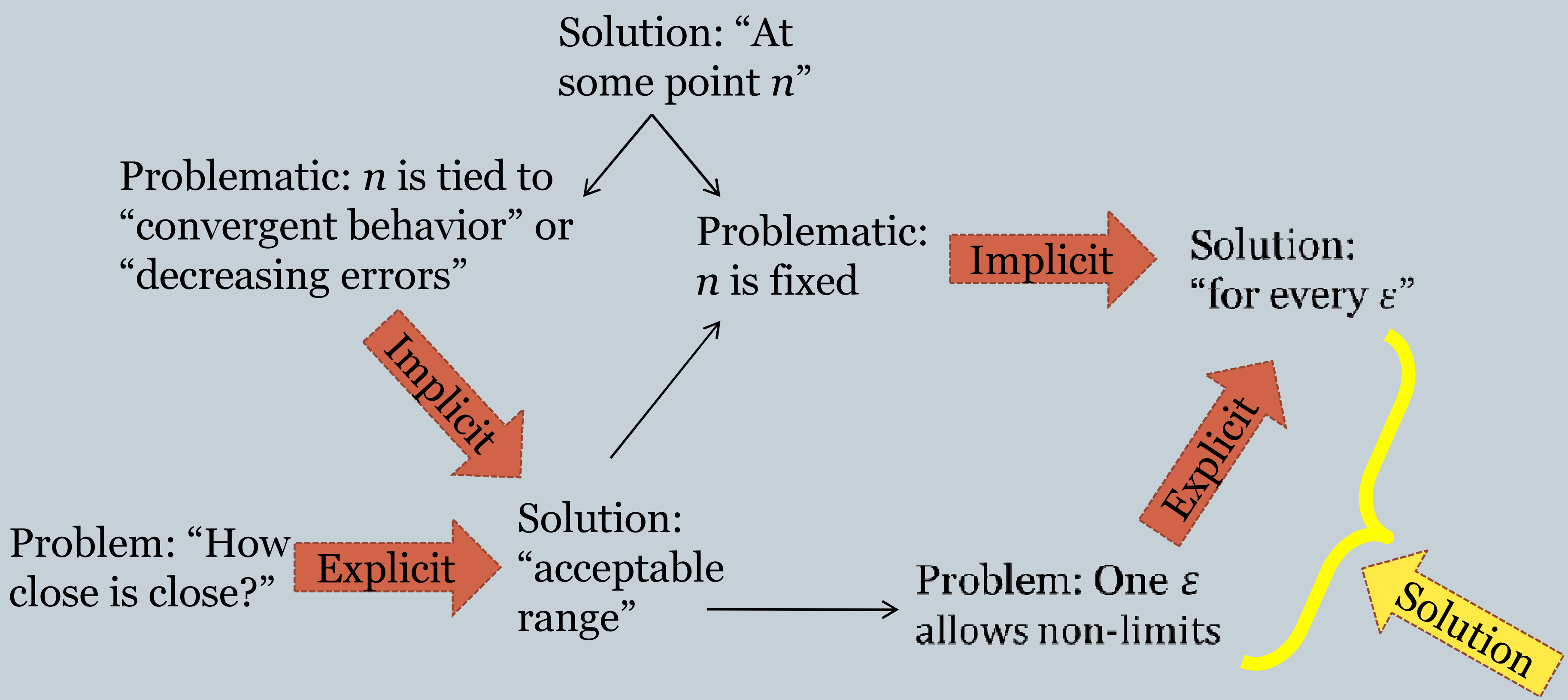
Megan: I guess we can say "within a designated margin of error" or "designated error bound" because it would vary by what our error bound was for that particular problem or situation.

(Day 1, 27:00)

Problem: "How close is close?" → Explicit → Solution: "acceptable range"

Well if an alternating [graph] never actually becomes 5 but we ***accept*** it as a convergent graph, um, that must mean at some point it gets within a range of 5 [*holds hands apart horizontally*] that we accept it as converging to 5 whereas say if we're looking at the sine graph [*moves hands farther apart*], it doesn't get within that, that distance. It stays out at –1 to 1…. The regular sine graph. It stays within –1 to 1. That distance is too great to call it convergent. But if [the alternating graph] never actually becomes 5 [*moves hands back together*], but it gets that small little range. There's gotta be a little acceptable range at some point that you say it's convergent [*holds two fingers close together horizontally*] even though it never actually becomes [*bounces fingers off to her right*].

Problems and Solutions

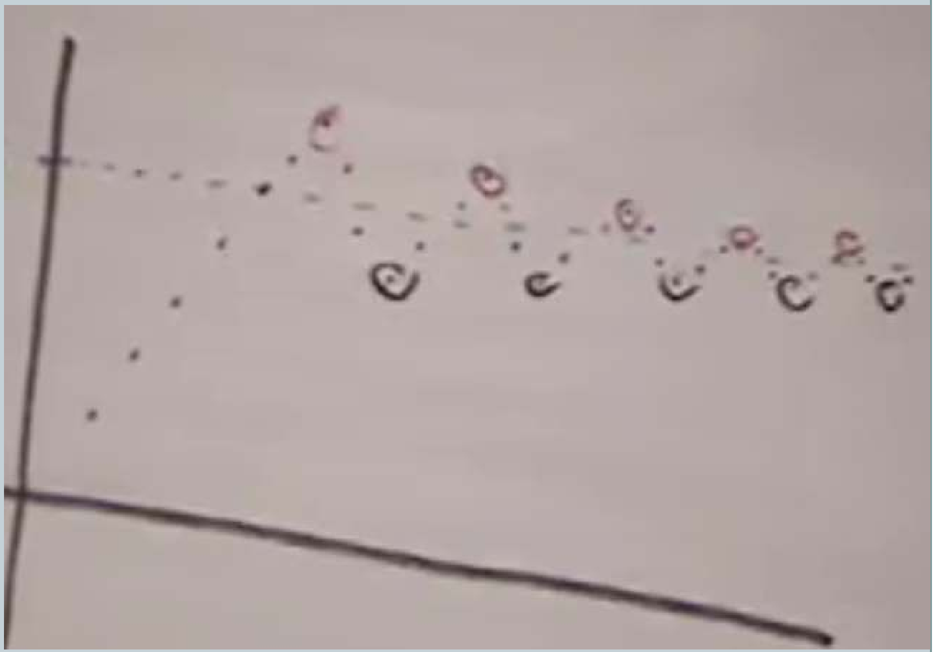


Questioning the “acceptable range”

- Its size is “arbitrary” (not intentionally or well-chosen)
- “Decreasing errors” captures the idea of “getting closer”
- Different numbers create different definitions
- One size doesn’t eliminate all non-limits
- It allows Graph E to converge

The “acceptable range” allows Graph E to converge

Megan: Because it will eventually get it within whatever range we want to put it in though, whether it’s point 25 zeros and a 1 or a thousand zeros and a 1, it will get within that range and it will stay there.



Problem: One ε allows non-limits → Explicit → Solution: "for every ε"

After wrestling with the problem for over an hour:

Interviewer: What about "for EVERY error?"

Belinda: So then that would work because ***every*** error, 'cause then it has to work within ***every*** error or it's not convergent. If it doesn't work within ***every*** error, then it's not convergent.

Megan: Yeah, that works.

A sequence converges to 5 as $n \to \infty$ only if there exists a value N after which $|5 - a_n|$ does not exceed every error after N.

Problem: One ε allows non-limits → Explicit → Solution: "for every ε"

- Appropriate timing for providing a solution

- They wrestled with the problem from several angles for about 1 hour.
- They had previously introduced the universal quantification for ε.

- They immediately seized on and applied the solution.
- Their universal quantification of ε persisted.

Problems and Solutions

Definition 3: As $n \to \infty$, at some n, a_{n+1} becomes closer to (or is) 5.

Problematic: *n* refers to multiple things

Problematic: which a_n's?

Solution: "At some point *n*"

Problematic: *n* is tied to "convergent behavior" or "decreasing errors"

Problematic: *n* is fixed

Problems and Solutions

Problem: n refers to multiple things

Explicit

Solution: N

Conflict

Problematic: n refers to multiple things

Problematic: which a_n's?

Solution: "At some point n"

Problem: n refers to multiple things → Explicit → Solution: N

Definition 13: "A sequence converges to 5 as $n \to \infty$ only if there exists a value n after which the $|5 - a_n|$ does not exceed your chosen error and $|5 - a_{n+1}| < |5 - a_n|$."

Problem: n refers to multiple things → **Explicit** → Solution: N

Craig: My question was this $5-a_n$, what does a_n refer to?... The reason I'm asking is because it occurs to me that we're using n to mean, given some error, a static place here, right?

Megan & Belinda: Mmm-hmm

Craig: And then this n refers to a whole bunch of points.

Megan: Yeah.

Craig: Um, could it clear up confusion to call one of them something else maybe?

Belinda: Maybe a big N for this?

Audience Poll

What best describes the role Craig played in the previous slide?

a) **Guide the Process:** Steer the ship toward productive waters

b) **Produce Conflict:** Ask questions designed to turn "Problematic Issues" into "Problems"

c) **Provide Solution:** At "appropriate" times, offer solutions to "Problems"

d) **Elicit Reasoning:** Ask for elaboration to produce rich data

Problem: n refers to multiple things → Explicit → Solution: N

Definition 14: A sequence converges to 5 as $n \to \infty$ only if there exists a value N after which the $|5 - a_n|$ does not exceed your chosen error after N and $|5 - a_{n+1}| < |5 - a_n|$.

Problems and Solutions

Problem: n refers to multiple things

Explicit

Solution: N

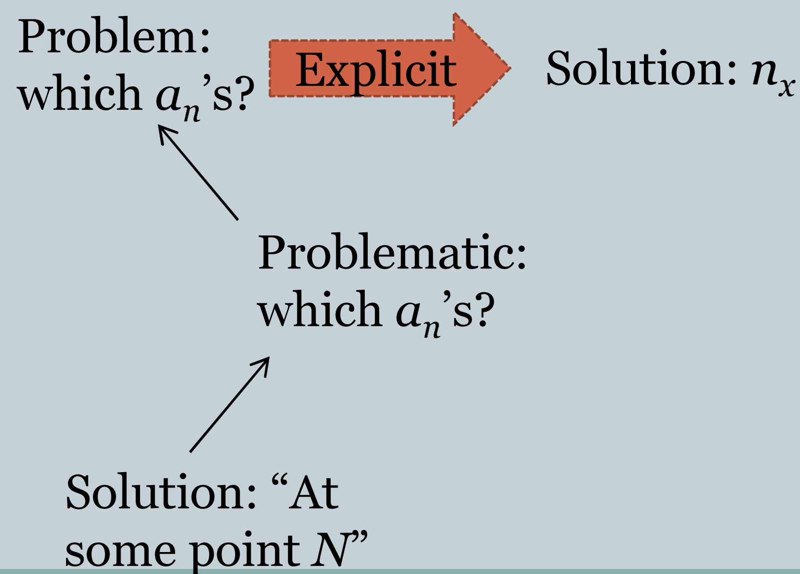
Conflict

Problematic: n refers to multiple things

Problematic: which a_n's?

Solution: "At some point N"

Problems and Solutions

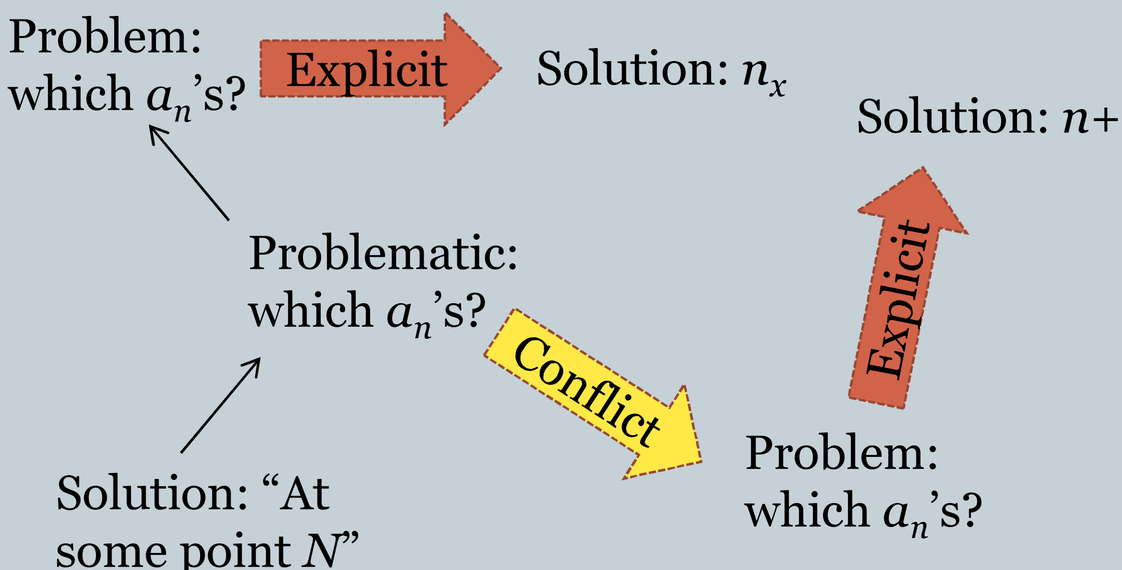


Problem: which a_n's? → Explicit → Solution: n_x

Definition 8b: "A sequence converges to 5 as $n \to \infty$ i.f.f. $\exists\ n_x$ such that for all a_{n_x} terms after $|5 - a_n| \leq .01$"

Megan: I don't know. Just trying to think of a way to designate that point *n* forward and neglecting what's going on before that."

Problems and Solutions



Problem: which a_n's? → Explicit → Solution: $n+$

Craig (aka, a Conflict Producer): What does a_n refer to?

Megan: That's the approximation.

Belinda: That's whatever the term is. The approximation….All to the right of n.

Craig: Oh but not these guys back [here], because these are a_n's too aren't they?

Megan & Belinda: Yes.

Craig: But you're just saying the a_n's over here.

Megan: Mmm-hmm

Belinda: So maybe if we put like a little plus sign on that n to show that it's like, to the right.

Problems and Solutions

Problematic: which a_n's?

Solution

Solution: $n \geq N$

Conflict

Solution: "At some point N"

Conflict → Problem: which a_n's? → Solution → Solution: $n \geq N$

Craig: [*pointing to* $|U\text{-}a_n|$ *in their definition*] You're saying that distance is within every error bound epsilon?

Belinda: Yeah.

Jason (aka, conflict producer): Is that true up there [*on the board*]?

Belinda: No, cause it's not true until you get to the N.

Megan: Yeah.

Belinda: So after some value N.

⋮

Craig (Solution Provider): In other words, this little n would all be bigger than N.

“A sequence converges to 5 as $n \to \infty$ provided…”

Definition 1: “A sequence converges to 5 as $n \to \infty$ provided $n + 1$ is closer to 5 than n for any n value.”

Two Days Later

Definition 23: “A sequence converges to U when $\forall \varepsilon > 0$ there exists some N $\forall n \geq N$ $|U - a_n| \leq \varepsilon$”

Lessons Learned

- Solutions to problems are useful
- Solutions to problems persist
- Solutions to problematic issues do neither

- Facilitators must act as Conflict Producers
  - Productivity is stalled
  - A particular problematic issue needs to be resolved
- Facilitators must act as Solution Providers
  - The students have wrestled with the problem
  - They have elements of the solution

Open Response

Are there elements of this teaching experiment that could be incorporated into calculus courses to help students make the transition to proof-based upper division courses?

(Please enter your response in the chat window.)

Questions and Comments

Thank you!