Using Angle Measure to Foster Connections in Trigonometry

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The research reported in this paper was supported by the National Science Foundation under grant number EHR-0412537. All opinions expressed are solely those of the authors.

What does it mean for an angle to have a measure of one degree?

Some Responses

- 2 lines are connected at one end and one of the lines is slanted away from the other by 1º.
- It means the angle is very acute and is 1º out of 180º away from not having any angle at all.
- If you drew an angle on the other side its measure would be 179º.
- It means it is an acute angle, very small. It also means that is 1/360 of a circle.
- This angle is acute. It is a very small angle. For this angle to be a part of a triangle, the sides would have to be very long.

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I'm not really sure what it means…

Survey of Textbooks

A common approach to angle measure:

- Define an angle and investigate in the context of shapes.
- Classify angles by their type.
- Use a protractor to measure the angle.
- Define and calculate supplementary, complementary, adjacent, interior, etc. angles.

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- **During what grades do angle, degree measure, and radian measure first appear in the Common Core State Standards?**

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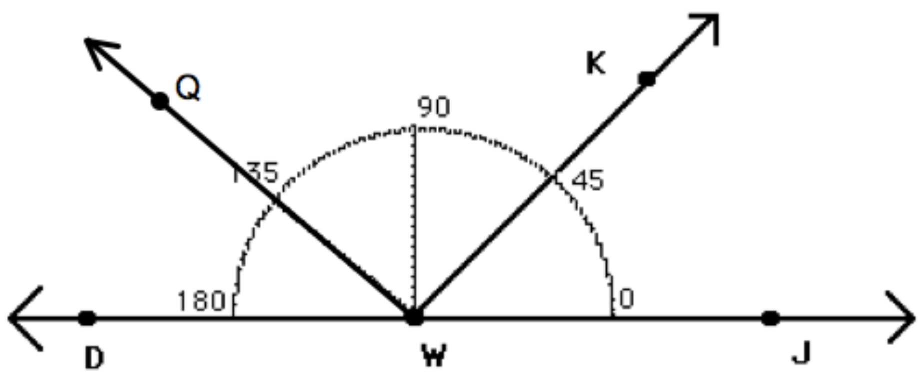
- Define and calculate supplementary, complementary, adjacent, interior, etc. angles.
- Common Core:
  - Angles: Grade 2.
  - Degree Angle Measure: Grade 4.
  - Radian Angle Measure: High School (Functions).

Protractor Postulate

Given line DJ and a point W on line DJ. Consider rays WJ and WD, as well as all the other rays that can be drawn, with W as an endpoint, on one side of line DJ. These rays can be paired with the real numbers between 0 and 180 in such a way that:

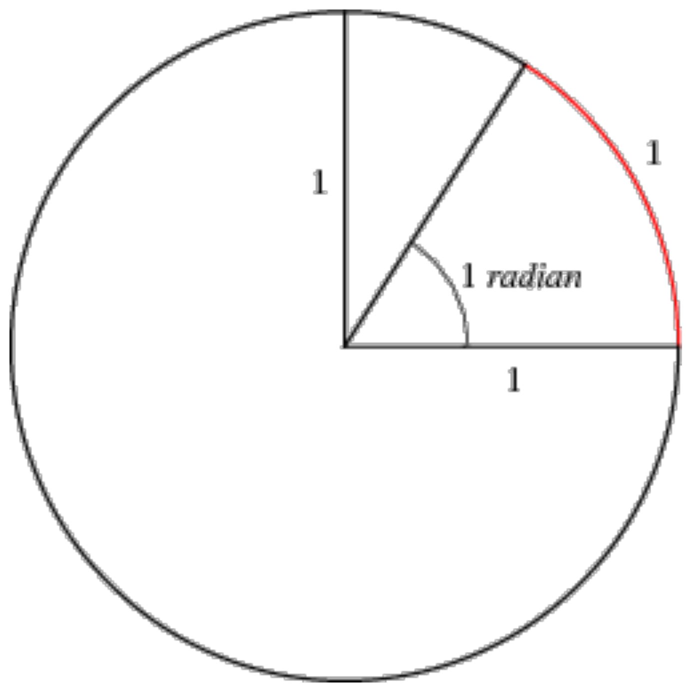
1. Ray WJ is paired with 0, and ray WD is paired with 180.
2. If ray WK is paired with *a* and ray WQ is paired with *b*, then the measure of angle QWK = | *a* - *b* |.



What does it mean for an angle to have a measure of one radian?

One Radian

An angle with a measure of one radian is subtended by an arc of one radius. A radian angle measure is determined by dividing the arc length cut-off by the corresponding radius.



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Research has shown that students and teachers have difficulty reasoning about trigonometric functions, often stemming from their understandings of topics foundational to trigonometry.

- Teachers tied to right triangles and trigonometry being about solving for sides of triangles (Thompson, Silverman, and Carlson, 2007).
- Degree angle measure dominates student conceptions of angle measure, tying their images of trig functions to right triangles (Fi, 2006).
- Students unable to give meaningful explanations of sine as a function (Weber, 2005).

So what's the big deal?

Research has shown that students and teachers have difficulty reasoning about trigonometric functions, a difficulty that stems from their understandings of topics foundational to trigonometry.

- Lack of ability to construct geometric objects to reason about or estimate trigonometric values (Weber, 2005).
- Students define the trig functions at positions, as opposed to on the real numbers (Akkoc, 2008; Brown, 2005).
- Students view π as a position or "unit" of measure and not as a real number (Akkoc, 2008; Fi, 2006).
- Students view angle measures as labels of geometric objects and lack a systematic process by which to measure an angle (Moore, 2009).

Complicating the issue…

A survey of the history of trigonometry and common US curricula approaches to trigonometry have revealed:

- Classroom approaches to angle measure and trigonometric functions stand in stark contrast to the history of these topics (Bressoud, 2010).
- There is a fundamental cognitive divide created by the contrasting approaches to degree and radian measure, even those both of these units measure the same quantity (Thompson, 2008).

Response

Designing an instructional sequence informed by a combination of the historical developments of trigonometry and mathematics education research (e.g., trigonometry, function, and quantitative reasoning).

Using a series of teaching experiments (Steffe & Thompson, 2000) to investigating student thinking (including pre-service secondary teachers) during the instructional sequence.

Instructional Sequence and Student Work

Pre-Interviews

Angle measure understandings rooted in labels of geometric objects (e.g., a circle is 360 degrees, a line is 180 degrees).

Lack a systematic means by which to measure an angle without a protractor.

When given an arc length, the students could calculate an angle measure but had difficulty justifying their calculations beyond pairing units.

Initial Goal

Create a situation in which the students' conceptions of angle measure would be brought to the forefront and cause a disequilibrium.

Develop a systematic method for measuring an angle that involves coordinating quantities' measures.

Construct common meanings for degree and radian angle measure that emerge from the students' actions.

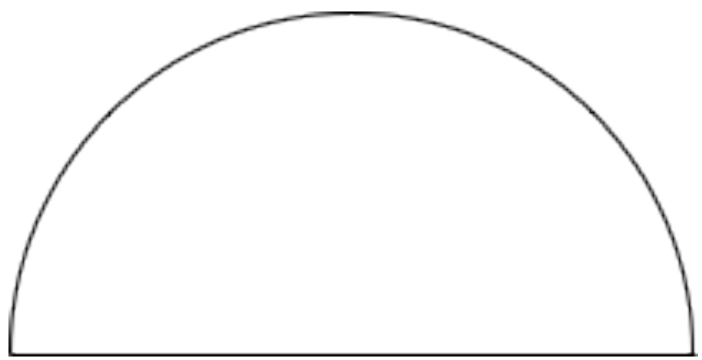
Protractor Activity

Supplies: Blank protractors, Wikki Stix, ruler

Let's say we want to measure an angle in a number of "gips" and we know 8 gips rotate a circle. Construct a protractor that measures an angle in a number of gips.

Let's say we want to measure an angle in a number of "quips" and we know 15 quips rotate a circle. Construct a protractor that measures an angle in a number of quips. Explain this construction.

How would you construct a protractor measured in degrees?



Protractor Activity

Students first used notions of area to "divide the protractor" into "equal pieces" and relate the pieces to degree angle measures.

This reasoning presented difficulties when creating a protractor to measure in quips, and subsequently caused the students to rethink their approach to the problem.

The students then identified an arc length per quip (or gip) that they could use to partition the circumference of the protractor.

Protractor Activity

As an example, for circles with circumferences of approximately 31.5 centimeters and 51 centimeters, the students determined arc lengths of approximately 2.1 centimeters and 3.4 centimeters per quip, respectively.

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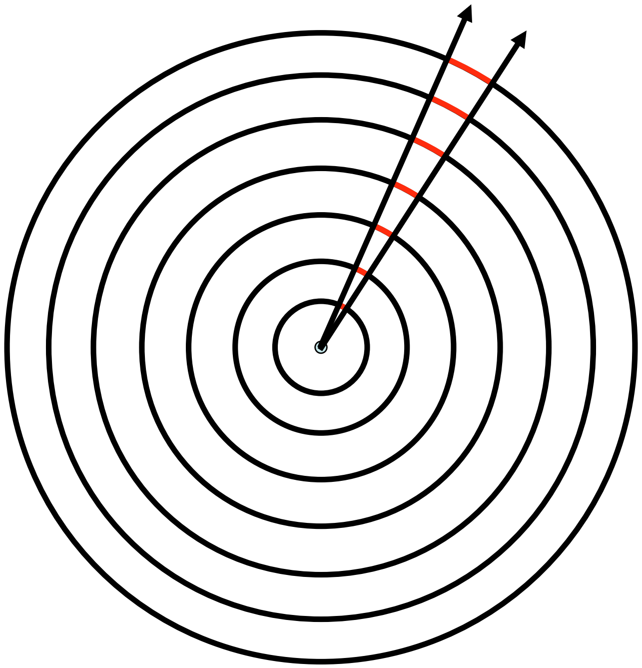
$$\frac{2.1}{31.5} = \frac{3.4}{51} = \frac{1}{15}$$

An angle of measure one quip is subtended by an arc that is one-fifteenth of the corresponding circle's circumference.

Protractor Activity

Extending to degree measure:

An angle measure of *d* degrees conveys that the angle is subtended by an arc that is *d*/360ths of the corresponding circle's circumference.



T or F: A radian measure is unitless.

Unitless?

T: It is the ratio of two lengths.

$$\frac{3.2\,ft}{2.1\,ft}$$

Unitless?

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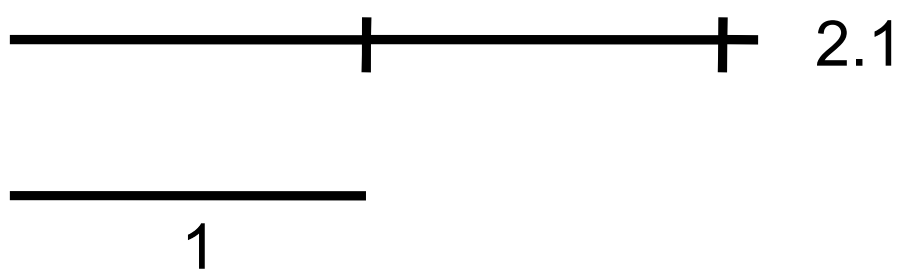

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This is more an artifact of the calculation used to make the measure (e.g., dividing the measure of an arc length in feet by the radius measured in feet).

In the same way, a measure in feet is the ratio of two lengths; a process of partitioning or forming a multiplicative comparison.

Goal

Tying radian measures to conceptualizing a circle's radius as a unit of measure.

Develop a systematic method for measuring an angle that involves coordinating quantities' measures (e.g., measuring along an arc).

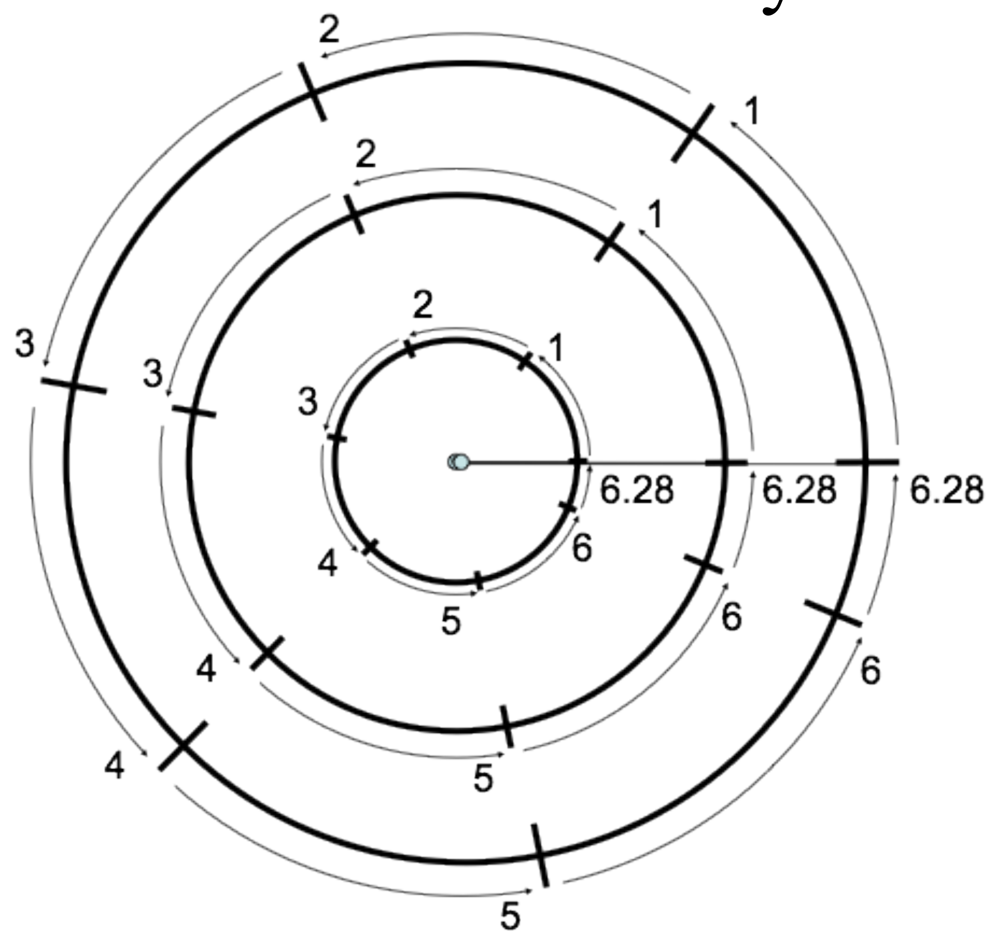
Construct common meanings for degree and radian angle measure that emerge from the students' actions.

Radius Activity

Each person within a group should have a different length Wikki Stix. Do the following on a blank sheet of paper:

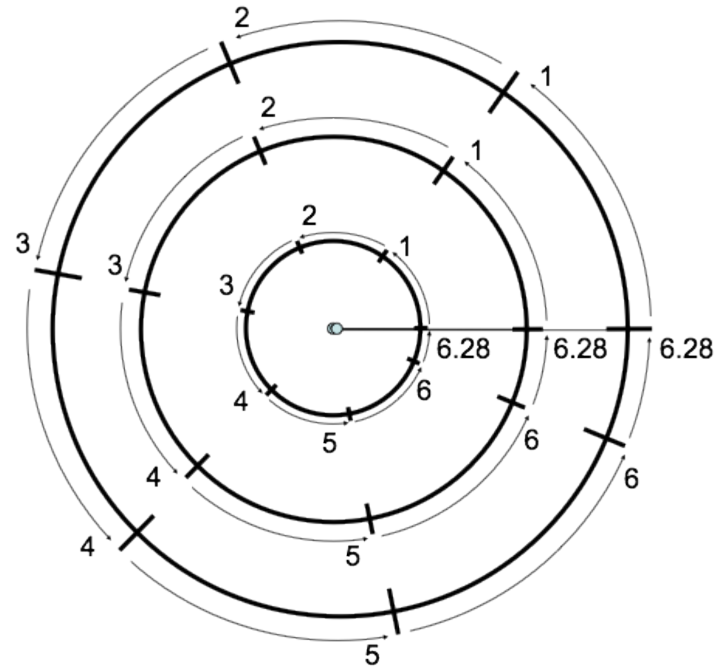
- Use the Wikki Stix and compass to create a circle with a radius the length of your Stix.
- How many Wikki Stix will fit on the circumference of your circle? Create an angle that:
  - is subtended by one Wikki Stix.
  - is subtended by 2.5 Wikki Stix.
  - is subtended by 7 Wikki Stix.
- Compare your results with your group members.

Radius Activity

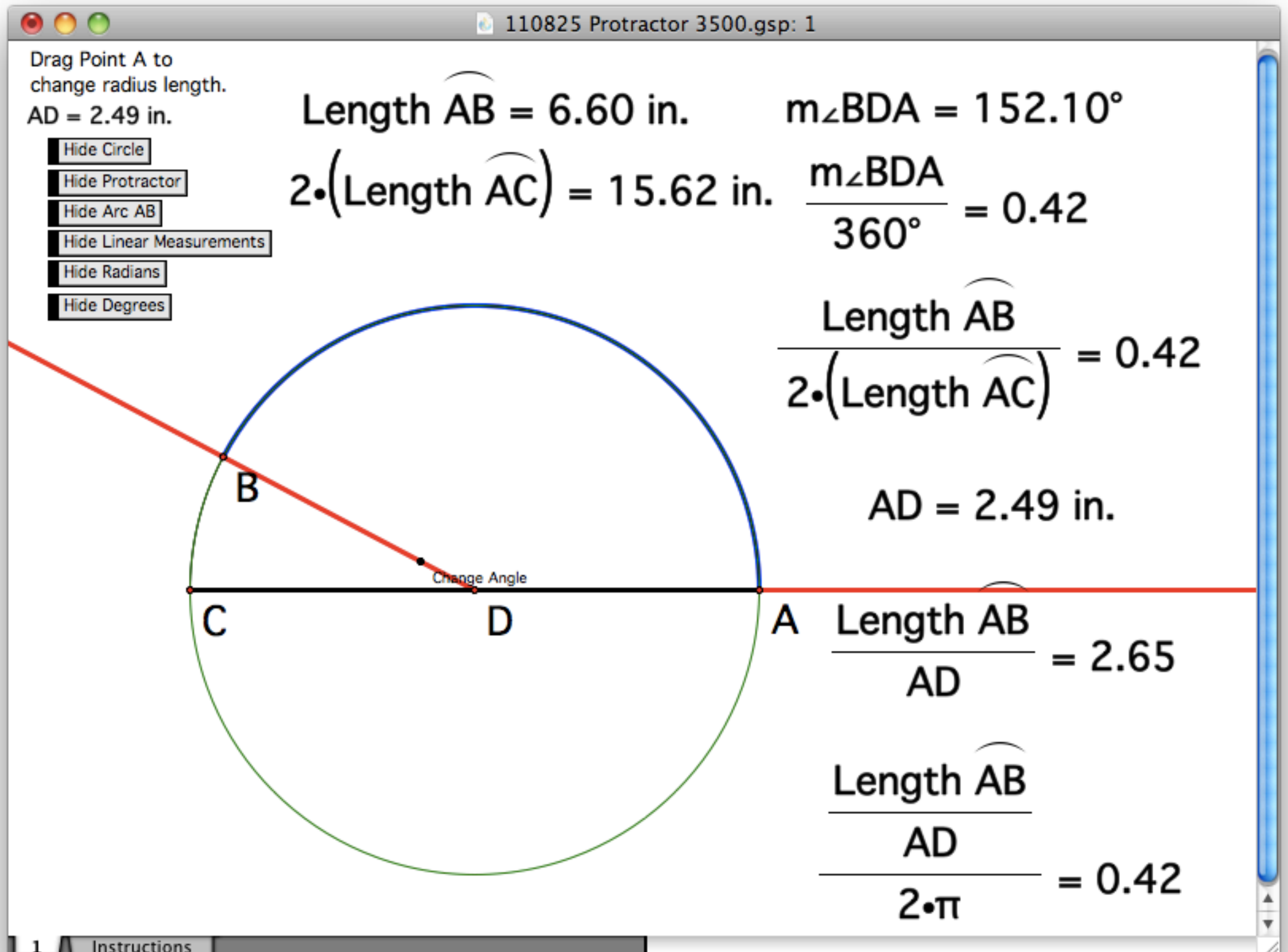


Radius Activity

An angle with an openness of r radians is subtended by an arc with a length of r radii. The angle can also be said to be subtended by an arc that is $r/(2\pi)$ of the corresponding circle's circumference.

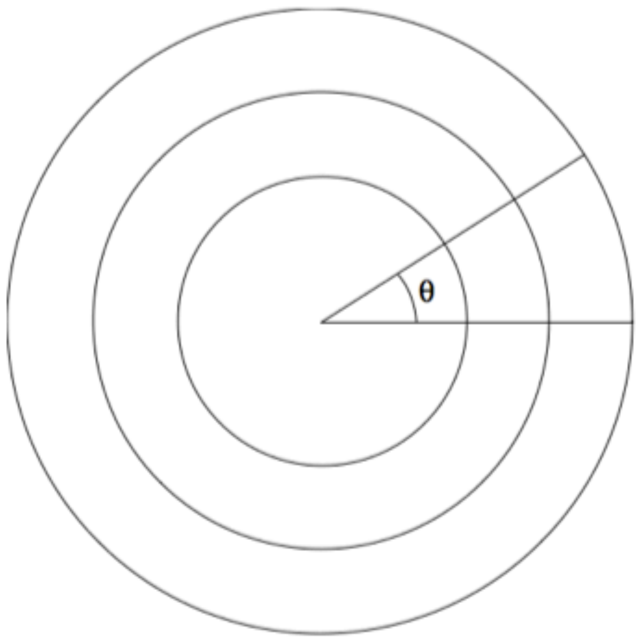


Any Radius…


110825 Protractor 3500.gsp: 1
Drag Point A to
change radius length.
AD = 2.49 in.
Hide Circle
Hide Protractor
Hide Arc AB
Hide Linear Measurements
Hide Radians
Hide Degrees
Length AB = 6.60 in.
2•(Length AC) = 15.62 in.
m∠BDA = 152.10°
m∠BDA / 360° = 0.42
Length AB / 2•(Length AC) = 0.42
AD = 2.49 in.
Length AB / AD = 2.65
(Length AB / AD) / 2•π = 0.42
B
C
D
A
Change Angle
1
Instructions

Outcomes

1. Given that the following angle measurement θ is 35 degrees, determine the length of each arc cut off by the angle. Consider the circles to have radius lengths of 2 inches, 2.4 inches, and 2.9 inches. (*Drawing not to scale*)



Outcomes

Predict a Solution

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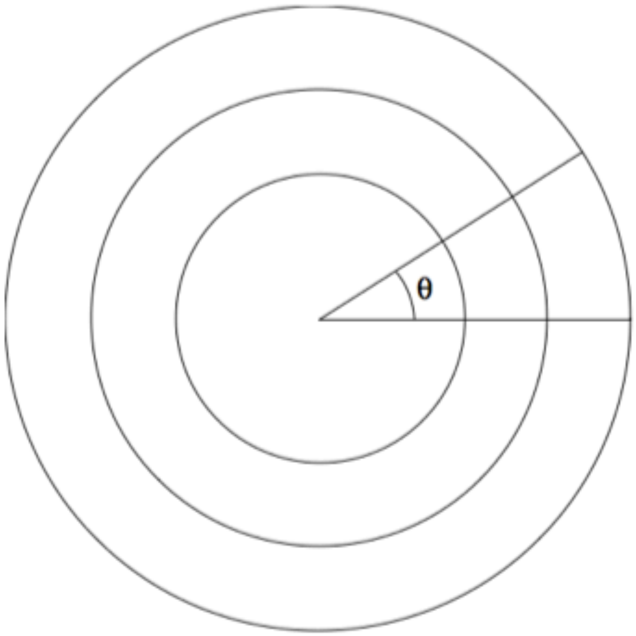
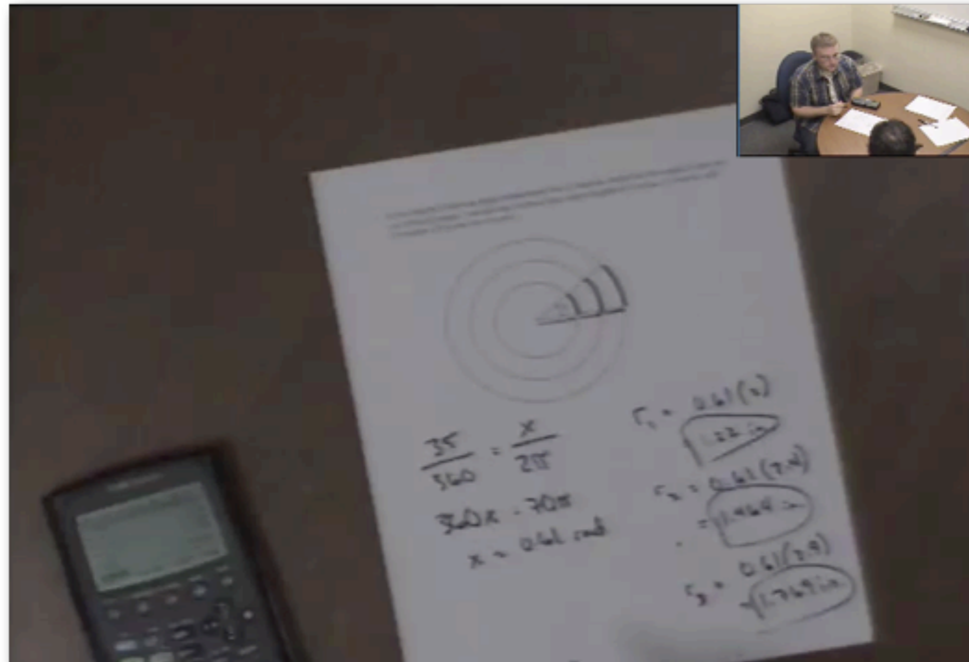
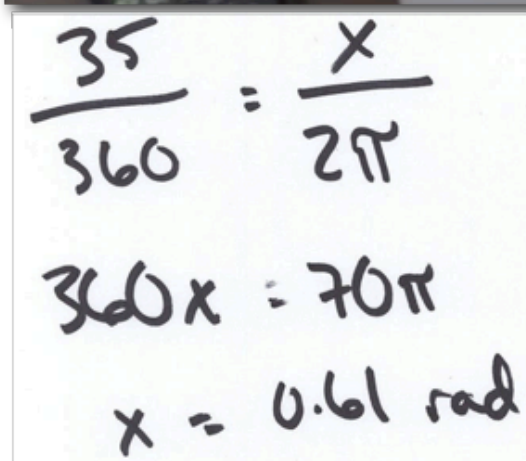
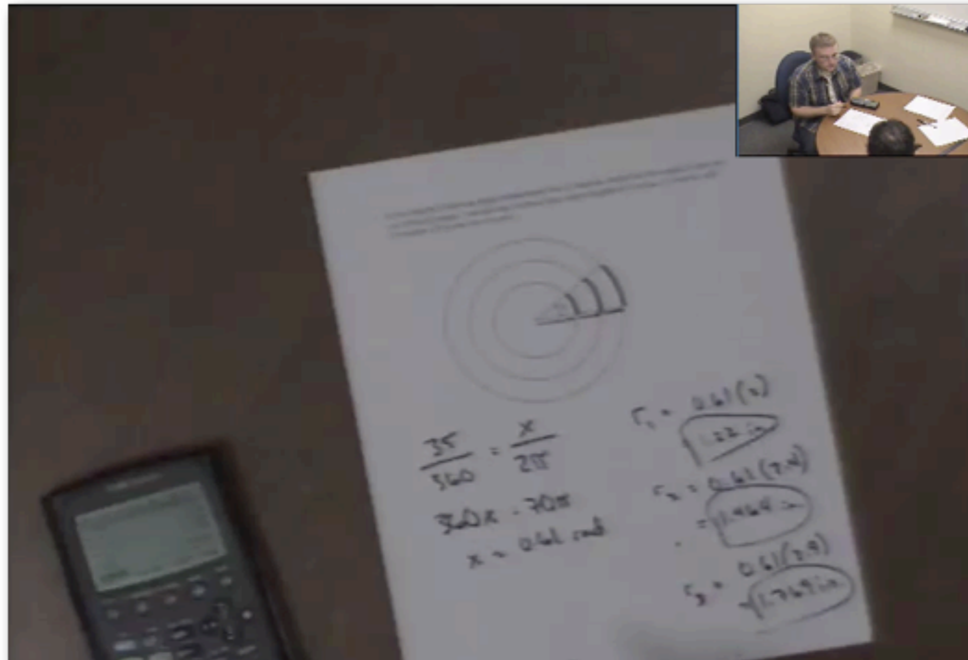


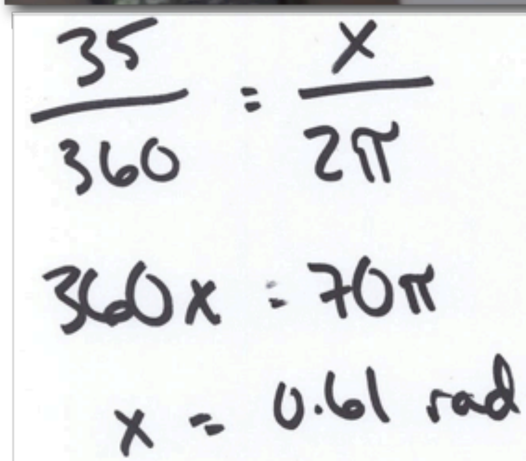
So what I plan on doing for this one is converting thirty five degrees into radians...So those two should be equal and I can just find x. And then with that all I have do is just multiply the answer by two inches, two point four inches, and two point nine inches to get the different arc lengths right there, because *radians is just a percentage of a radius.*





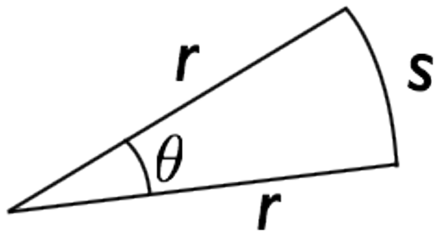
What you're doing is just finding a percentage. Like thirty-five over three-sixty is (*using calculator*), is poi-, or nine point seven percent of the full circumference. 'Cause three hundred sixty degrees in a circle...So thirty-five of those degrees equals nine point seven percent of the full thing. Then so all I have to do is multiply nine point seven percent by two pi, and I'll find x.





Outcomes

$$\theta = \frac{s}{r} \quad \Big| \quad r\theta = s$$



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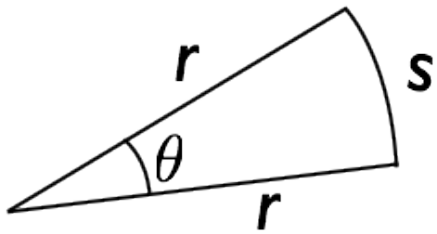


Alright. We'll say theta equals radians, very very simple then. *r* theta is equal to *s* (*writing*). 'Cause theta is in radians, that means a percentage of the radius. Which would then be equal to this length (*tracing arc length*). So you multiply the percentage of the radius by the radius you'll get the arc length.

Well this is…a percentage of a radius length over a radius…a ratio, that'll give you a percentage of *r*.

Moving into Trigonometric Functions

Developing Trigonometric Functions by Modeling Circular Motion

Leverage the students' arc length images of angle measure.

Draw on reasoning about the radius as a unit of measure to generate the unit circle.

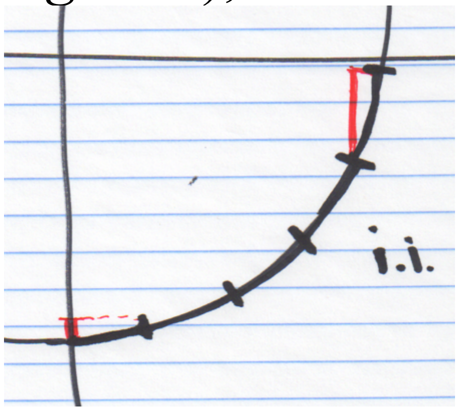
Use rate of change reasoning to construct the sine function.

Example

Consider a Ferris wheel with a radius of 36 feet. April boards the Ferris wheel at the bottom. Determine a graph and formula that relates April's total distance traveled along the path formed by her trip and her vertical distance from the boarding position.

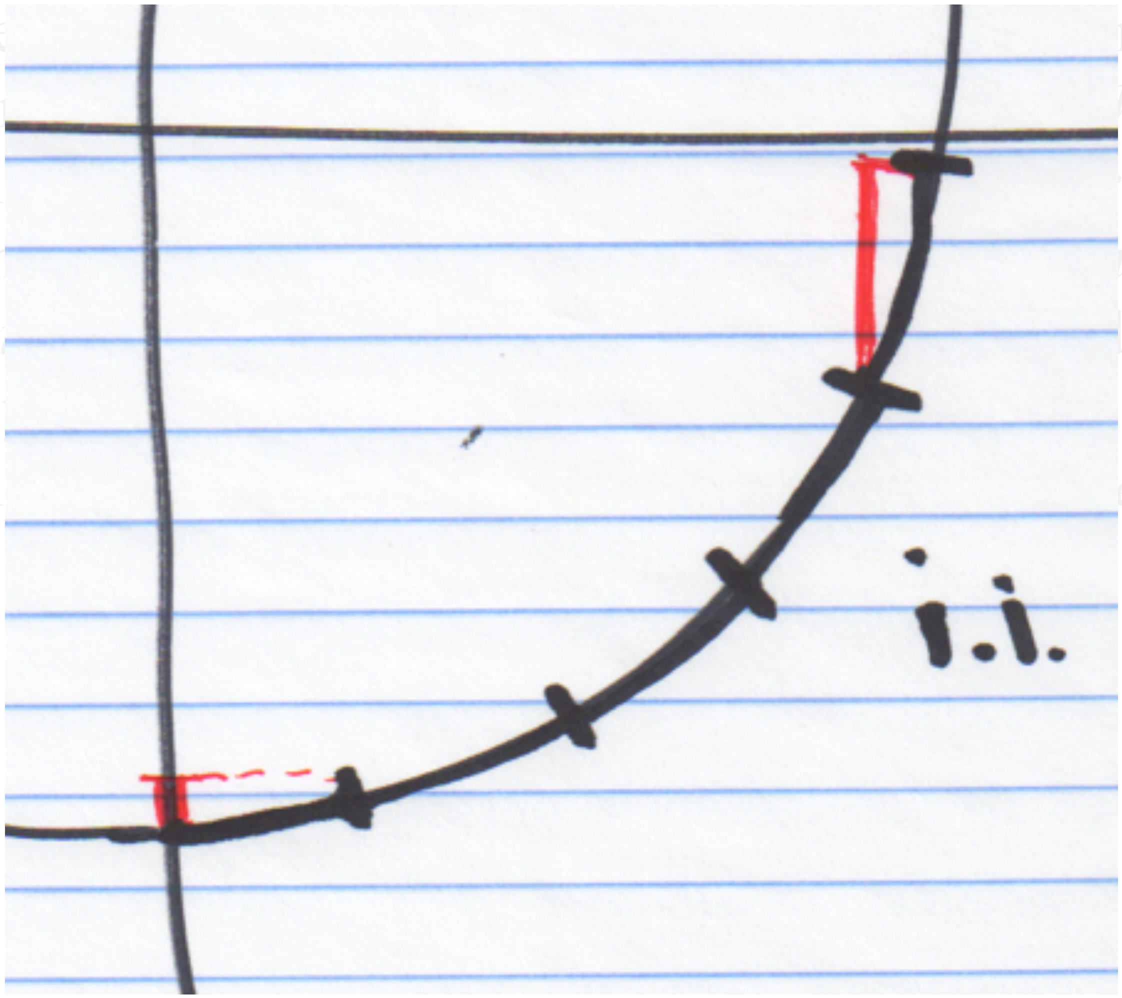
Student Solution

Ok...we're here (*pointing to starting position*), for every unit the total distance goes (*tracing successive equal arc lengths*), the vertical distance is increasing at an increasing rate (*writing i.i.*)...she moves that much there (*tracing an arc length*), that much here (*tracing an arc of equal length*), uh, the vertical distance there changes by that much (*tracing vertical segment*). And then, uh, the vertical distance here changes by that much (*tracing vertical segment*), which is a much bigger change.



Student Solution

Ok...we're ... ry unit the
total dista... *hs*), the
vertical di... *iting*
i.i.)...she r...), that much
here (*trac*... istance there
changes b... then, uh, the
vertical di... *ertical*
segment), ...



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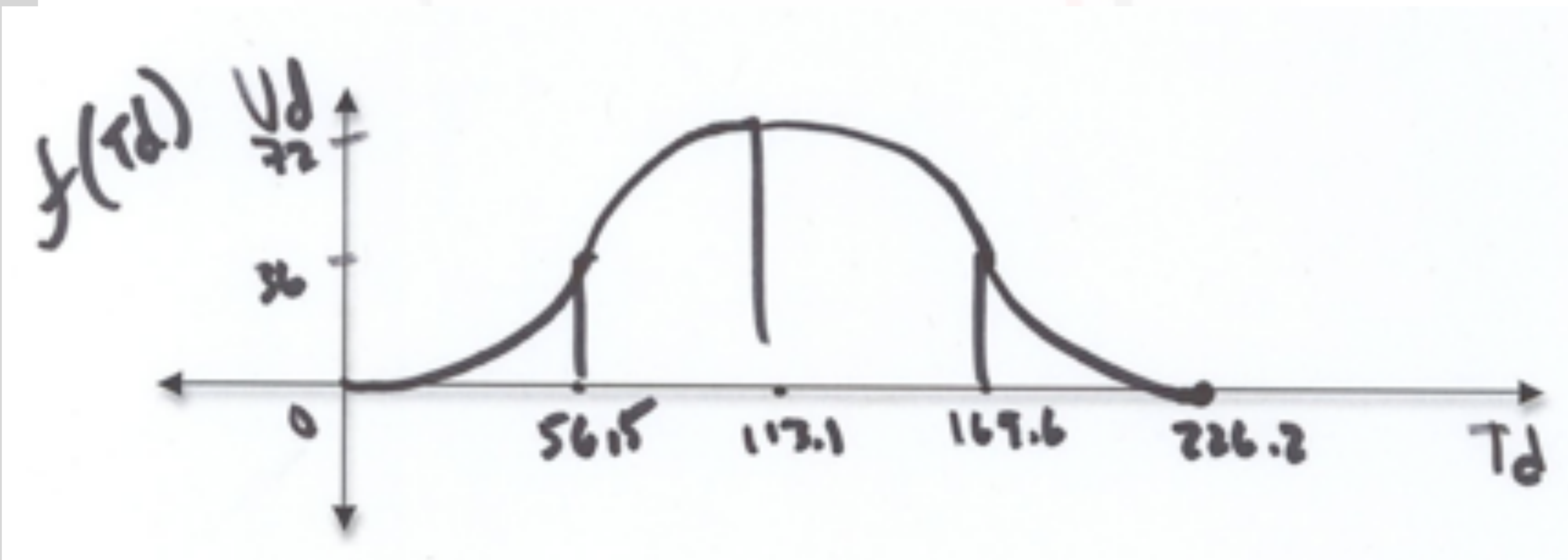

Student Solution

$$\frac{Td - 56.5}{36} \quad \Bigg| \quad Vd = f(Td) = \sin\left(\frac{Td - 56.5}{36}\right)$$

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$$Vd = f(Td) = 36\sin\left(\frac{Td - 56.5}{36}\right)$$

After prompting from the interviewer to explain "vertical distance," Zac identified a vertical distance from the horizontal diameter of the Ferris wheel.

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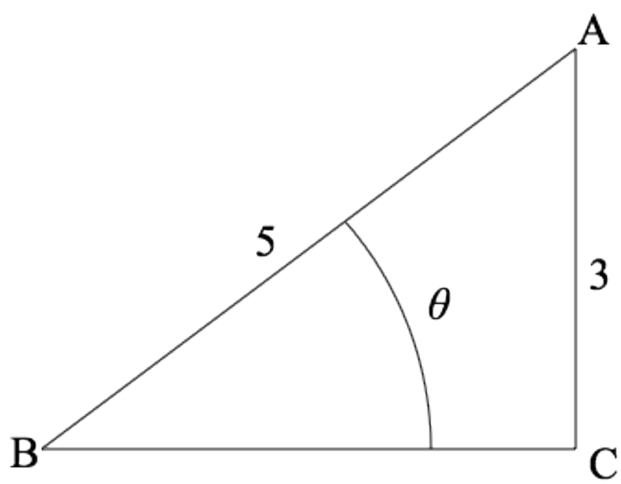
$$Vd = f(Td) = 36\sin\left(\frac{Td - 56.5}{36}\right) + 36$$

Moving into Right Triangles

Students were asked to relate the previously learned trigonometric ratios in a right triangle context to their previous explorations.



5. While site seeing in New York City, Bob stopped 1000 feet from the Empire State Building and looked up to see the top of the Building. Given that the angle of Bob's site from the ground was 56 degrees, determine the height of the Empire State Building.

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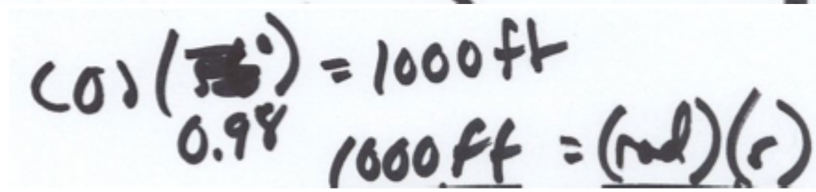

"From the circle, or triangle, we can determine that cosine of fifty six degrees is equal to one thousand feet (writing corresponding equation)…one thousand feet is equal to the radians, because cosine fifty six degrees is determined in radius lengths."

5. While site seeing in New York City, Bob stopped 1000 feet from the Empire State Building and looked up to see the top of the Building. Given that the angle of Bob's site from the ground was 56 degrees, determine the height of the Empire State Building.

I always just thought hypotenuse was, you know, just that side of a triangle. You could use Pythagorean's Theorem to find out what it was very easily. Now I'm looking at it and seeing **it's the radius**, it makes a lot more sense to be able to find, you know, **the horizontal and vertical distance according to the radius.**

It simplifies the circle…using it as an actual unit… one radius, and then the circumference is six point two eight radius.

Summary

Notions of angle measure rooted in measuring *along* an arc:

-Offered connections between degree and radian angle measure.

- Reasoning about angle measure as a fraction of a circle's circumference.

-Created a foundation for the construction of the sine and cosine functions.

Students that continued to reason about angle measures as positions encountered difficulty progressing during the instructional sequence.

Summary

Students that continued to reason about angle measures as positions encountered difficulty progressing during the instructional sequence.

When asked to explain cos(π)=-1, one student explained that the equation referred to the bottom half of a circle.

Summary

Conceptualizing the radius as a unit of measure:

- Offered a tool of reasoning for the students to apply in novel situations.
- Resulted in the unit circle stemming from a process of enacting the radius as a unit of measure.
- Led to students generating spontaneous (and meaningful) connections between the unit circle and right triangles.

Summary

These ideas, in combination with rate of change reasoning, led to formulas and graphs as representations of covarying quantities' values.

The students valued these systems of meanings, particularly in the context of their previous experiences in trigonometry.

Looking Forward

Currently exploring students' construction of the polar plane in the context of their trigonometric understandings.

Initial data suggests that flexible notions of radian angle measure support students' spatial reasoning in the polar plane.

Also offering insights into imagining and representing covarying quantities' values.

Looking Backward

Work with pre-service secondary teachers is identifying unit conversion as critical for trigonometry.

-Basing conversions on unit cancellation restrict their use of the unit circle.

Students viewed the unit circle independent of other circles until they were asked to reason about how the measure of a quantity changes as the magnitude of the unit used to measure the quantity changes.

Looking Backward

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If my radius is 3.1 feet, and I make the radius a unit of measure, then all measures in radii are 1/3.1 times as large as the measures in feet.

Looking Backward

It is important that we find ways to create coherent systems of meanings that apply longitudinally in mathematics education, and then leverage these meanings in our teaching.

Trigonometry (*sigh*)

Trigonometry (*sigh*)

Meaningful doesn't imply easy and easy doesn't imply meaningful.

Thank You
and
Questions

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Some materials available at rationalreasoning.net