After 300 years …

A Significant New Look

at Quadratic Functions

$$f(x) = ax^2 + bx + c$$

Introducing …

The Migration Method

A new approach to graphing quadratic functions and determining its zeroes.

developed by John W. Coburn
of
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Some background …

Although the concept of fixing the location of a point using suitable coordinates dates back to antiquity, it is generally agreed that it was René Descartes (c. 1637) who formalized the connection between a function and its graphical representation in the plane.

Current methods …

Current methods for graphing a quadratic function involve evaluating the function for numerous inputs

(point-plot method)

or re-writing the function to consider the rigid and non-rigid transformations

(method of completing the square).

Although steeped in history …

Although steeped in history, protected by tradition, and proven over time, the value of these current methods is limited by the following (often inevitable) considerations:

① Each is time consuming

The time required to
a) repeatedly evaluate the function
or
b) complete the square
tends to limit the number of problems that can be completed and thereby the real-world connections that might otherwise be made.

Sign errors are common

$f(x) = x^2 - 5x - 24$

$f(-2) = -2^2 - 5(-2) - 24$

$f(-2) = -4 + 10 - 24$

$f(-2) = -18$ ***X***

❼ Procedural errors abound

$f(x) = x^2 + 8x - 5$

$f(x) = (x^2 + 8x + 16) - 5$

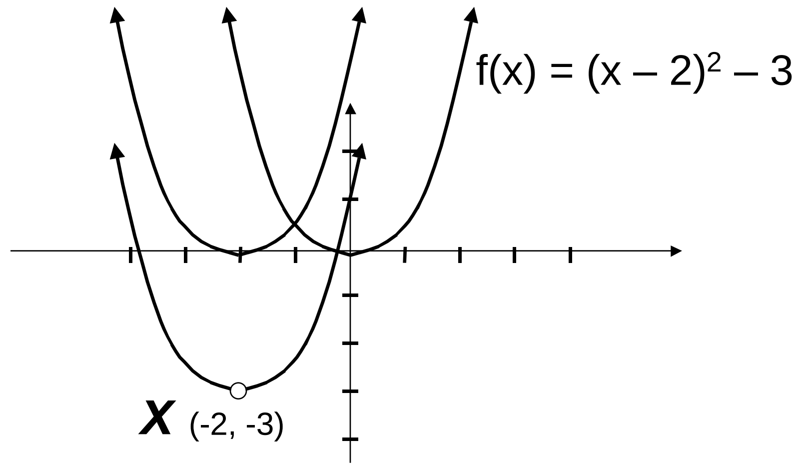
$f(x) = (x + 4)^2 - 5$ ***X***

$g(x) = 2x^2 - 12x - 7$

$g(x) = 2(x^2 - 6x + \_\_) - 7$

$g(x) = 2(x^2 - 6x + 9) - 7 - 9$ ***X***

Conceptual errors are common



🕮 Dependence on “nice” coefficients

$$f(x) = 2x^2 + 7x - 2$$

$$f(x) = 2(x^2 + ?x + \_\_) - 2$$

But perhaps the greatest limitation of these methods is ***the absence of a quick and efficient way to find the x-intercepts.***

Enter the Migration Method …

Although not a cure-all, the ***Migration Method*** (developed 11/00) goes a long, long way towards overcoming these "limitations," and contains an elegant simplicity that makes all aspects of the quadratic function more "accessible."

Here is the key …

half
square
times (- a)

Consider the function …

Consider the function defined by

$f(x) = x^2 – 10x + 13$ and its ***base function***

F(x) → the function less it's constant term.

$f(x) = ax^2 + bx + c \rightarrow F(x) = ax^2 + bx$

$f(x) = x^2 – 10x + 13 \rightarrow F(x) = x^2 – 10x$

Note that $a = 1$ and $c = 13$.

Now let's graph F(x) …

$F(x) = x(x – 10)$

The x intercepts are (0, 0) and (10, 0).

half of 10 → 5

square of 5 → 25

make negative → – 25

The vertex of F is (5, -25)!!

Migrate **F** to obtain the graph of **f** …

Migrate?

Yes – a vertical shift of "c" units (c = 13).

F:	(0, 0)	(10, 0)	(5, -25)
f:	(0, 13)	(10, 13)	(5, -12)

Just "connect the dots."

But what if b isn't even, say b = 7?

half of 7 $\rightarrow$ 3.5

For any digit **z**

$\mathbf{z}.5^2 = \mathbf{z}(\mathbf{z} + \mathbf{1}).25$

square of 3.5 $\rightarrow$ 12.25

make negative $\rightarrow$ **-**12.25

Consider $f(x) = x^2 - 7x - 5$ …

Consider the function defined by

$f(x) = x^2 - 7x - 5$ and its ***base function***

F(x) → the function less it's constant term.

$f(x) = ax^2 + bx + c \rightarrow F(x) = ax^2 + bx$

$f(x) = x^2 - 7x - 5 \rightarrow$ **$F(x) = x^2 - 7x$**

Note that a = 1 and c = - 5.

Now let's graph F(x) …

$F(x) = x(x - 7)$

The x intercepts are (0, 0) and (7, 0).

half of 7 → 3.5

square of 3.5 → 12.25

make negative → - 12.25

The vertex of F is (3.5, - 12.25).

Migrate **F** to obtain the graph of **f** …

Migrate?

Yes – a vertical shift of “c” units (c = - 5).

F:	(0, 0)	(7, 0)	(3.5, - 12.25)
f:	(0, - 5)	(7, - 5)	(3.5, - 17.25)

Just “connect the dots.”

But what if a ≠ 1 …

Consider the function defined by

$f(x) = 2x^2 + 7x – 10$ and its ***base function***

F(x) → the function less it’s constant term.

$$f(x) = ax^2 + bx + c \rightarrow F(x) = ax^2 + bx$$

$f(x) = 2x^2 + 7x – 10 \rightarrow$ $\mathbf{F(x) = 2x^2 + 7x}$

Note that a = 2 and c = - 10.

Now let's graph F(x) = x(2x + 7)

The x intercepts are (0, 0) and (- , 0).

half of - → -

square of - →

double

make negative →

The vertex of F is (- ,).

Migrate **F** to obtain the graph of **f** …

Migrate?

Yes – a vertical shift of "c" units (c = - 10).

F:	(0, 0)	(- 3.5, 0)	(- 1.75, - 6.125)
f:	(0, - 10)	(- 3.5, - 10)	(- 1.75, - 16.125)

Just "connect the dots."

Wow!

It is ***significant*** to note that the vertex of both the base function and the original function can now be determined ***using only elementary operations on the single value h***, since $k_o = -ah^2$ and $k = k_o + c$.

The excitement mounts …

This is all the more significant ***since h can be found without appealing to any formula or procedure***:

It is the midpoint of the x intercepts!

But what about the x intercepts of **f**?

The x intercepts are needed for a “true” and complete graph of **f**, and are often required for the application at hand.

The quadratic equation …

A student’s use of the quadratic equation is often hampered by the same considerations mentioned earlier: time, sign errors, procedural errors, conceptual errors, and so on.

$$x = \frac{\quad \sqrt{\quad\quad}}{\quad\quad}$$

Some background …

Methods for ***solving*** certain types of quadratic equations were developed by almost every major civilization and a general solution has been known for over a millennia.

Al-Khowârzmî's "formula" …

(circa 820 A.D.)

For $x^2 + px = q$

$$x = \sqrt{\; - \qquad} \qquad -$$

yields one solution.

Sridhara’s Method (circa 1025 A.D.)

$ax^2 + bx + c = 0$

$ax^2 + bx = -c$

(multiply by 4a and add b^2)

$4a^2x^2 + 4abx + b^2 = b^2 – 4ac$

$(2ax + b)^2 = b^2 – 4ac$

$x = \frac{\quad \sqrt{\quad}}{\quad}$

Vieta’s method (circa 1540 A.D.)

$ax^2 + bx + c = 0$

substitute $x = y – \frac{}{}$

Then expand and simplify to obtain:

$ay^2 + \frac{}{}$

Solve for y and subtract $\frac{}{}$

Harriot's Method (circa 1630 A.D.)

$ax^2 + bx + c = 0$

$x^2 +$ – x + – = 0

If r and s are roots, then …

$(x - r)(x - s) = 0$

$x^2 - (r + s)x + rs = 0$

Yielding r + s = - – and rs = –

Harriot's Method (cont.)

Using the relationship

$(r - s)^2 = (r + s)^2 - 4rs$

yields (after the appropriate substitutions)

$r - s = \frac{\sqrt{\quad}}{\quad}$

Solving the system containing r + s = - –

and r – s as shown, produces the quadratic formula.

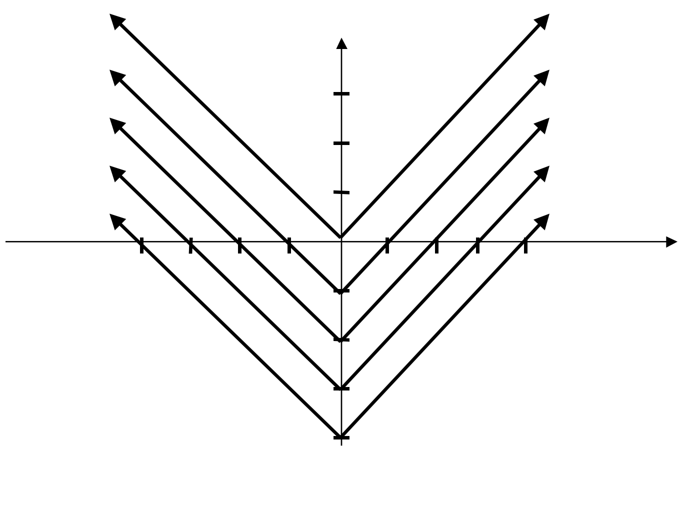
Here are the choices …

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

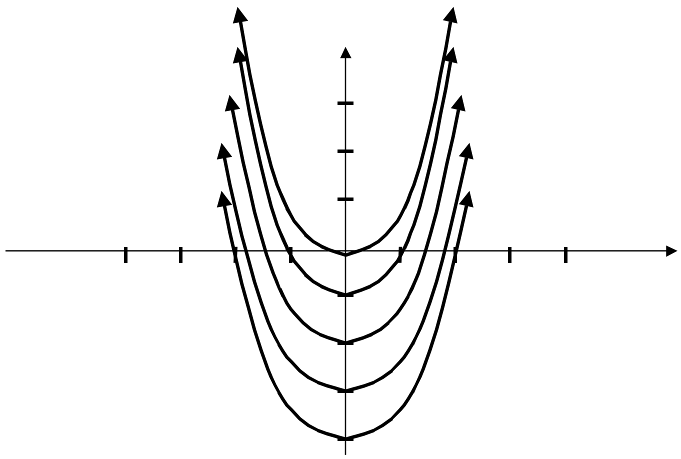
OR

$$x = h \pm \sqrt{-\frac{k}{a}}$$

f(x) = | x | ± k



$f(x) = x^2 \pm k$



Fortune smiles …

Claim: The x-intercepts (zeroes) of **f** are

$$x = h \pm \sqrt{-\frac{k}{a}}$$

(see overhead)

The magic of the method …

The x-intercepts of a quadratic graph can now be found using information easily obtained or already at hand. Namely the leading coefficient **a**, the mid-point **h**, and the y coordinate of the vertex **k**.

There's still more …

In addition, the method includes a direct reference to ideas which are very conceptual, but now very visible to our students. Namely, that if **a** and **k** have like signs …

the function will have no real zeroes!

Richard's Formula

$$x = h \pm \sqrt{-\frac{k}{a}}$$

I've named this new formula for solving a quadratic ***Richard's Formula***, in honor of my father. As an instructor of mathematics, there were few better.

In summary …

The efficiency of ***The Migration Method and the significance of its simplicity cannot be over-stated –*** as it circumvents the need to manipulate the function, encourages visual skills, eliminates the (student's) difficulty in using the quadratic formula, and enhances an understanding of both symmetry and shifts.

When used in conjunction with ***Richard's Formula***, all aspects of the quadratic function become more accessible, with no compromise in the integrity of the curriculum and no damage to the science or art of mathematics.

I hope you'll agree.

Thank you for coming, and please take a hand-out which explains and justifies ***The Migration Method*** and ***Richard's Formula*** in their entirety.