Appendix: Cribbage anyone? Ways to have “15 for 2” with 5 or fewer cards.

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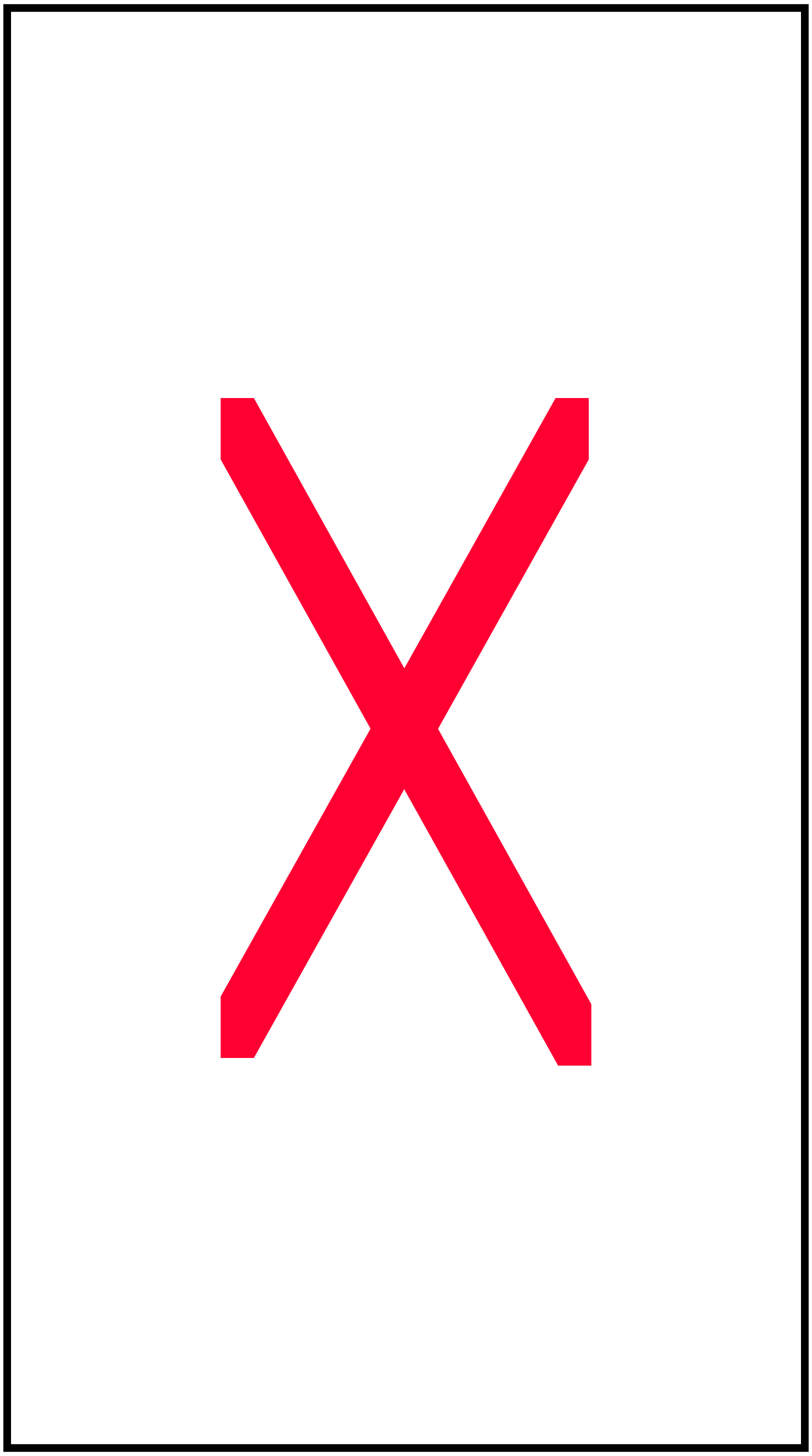

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How many ways can you...?

In cribbage, each different pair ("two of a kind") in your hand is worth two points. All score values for n-of-a-kind are derived logically from this fact. Three of a kind--being 3 = '3 choose 2' pairs--is worth six points. It doesn't take long to demonstrate physically with the cards the three "different" pairs in any given three of a kind. Similarly, since we find six different pairs in any four of a kind--the '4 choose 2' = 6 pairs of cards can be visual identified as different from each other--the value of four of a kind in cribbage is twelve. Basic scoring in cribbage requires skill (and drill) with combinations and permutations!

You get two points in cribbage for each combination totaling 15. Since combinations with a sum of 15 are important to the game, it was reasonable to ask: "How many different ways can you use five cards or less to get a sum of 15? List all of the numeric combinations involved." These small numbers do not lead to a trivial solution, but do lead students to want a systematic approach to sort and list their results. To check their work students first compared their (often chaotic) results directly--a tedious way to begin to ensure they "found them all, but didn't repeat any". A popular strategy to gain dependable results in the face of what seems like overwhelming complexity follows.

"separate and conquer"

Two systematic approaches to sort the details of the problem are duals of each other, the two approaches utilize the same strategy. Both require that the problem be methodically broken into several similar but simpler problems. Both yield partial solutions which are disjoint by design, so can be compiled conveniently through a simple union to construct the desired result. Both approaches reduce the tedium tremendously and lead to a feeling of assurance about the result. (And since they both yield the same result, we can feel secure that it was correct.) Both first demand an analysis of the question's parameters.

The sum was to be generated by cards. The cards were given value according to the number of pips on them, all picture cards were worth 10. (This is one of the random variables used in cribbage.) There were four cards available for each value from one to nine, and sixteen cards worth 10. Since a sum of 15 was required, every entry in our result would include at least two cards; the statement of the problem limited entries to at most five cards. Our final list would include only sets with two to five elements, the elements being numbers between 1 and 10 inclusive.

Identify two major concerns immediately—the distinction between cards and their values. First, we need to compile a list of all possible different combinations of numbers that add to fifteen (given the above constraints). Then, for each of these individually, we need to determine the ways the deck can provide for such a combination. With the first task completed, the second is mechanical. The first demands closer attention, and motivates a popular strategy for sorting.

keep as much as constant as possible: sort and count

The process requires two permeating guidelines. First, one choice is to work from large values down to small values, in the smallest increments possible. (Small values to large is also a viable option, with an analogous approach.) Second, always keep choices as constant as possible for as long as possible, before trading them in for the next smaller value. The one large problem was reduced to several steps--each being a restatement of the original question with a particular condition tacked on.

Stage one

Stage one: "How many different ways can you use five or fewer card-values to get a sum of 15--where the largest value is a 10? List them all."

Since one of the values is a 10, then we need consider other values only of 5 and under to fulfill stage one. The guideline was decided to start high and work methodically down, so {10,5} is first. The 5 can be reduced, so {10, 4,1} follows. Here the 1 can't be reduced, so move over and reduce the 4 and fill-in with the largest possible value to get {10,3,2}. By the stated strategy we want to keep the large values constant as long as possible, so skip the 3 for now and reduce the 2 to get {10,3,1,1}. The ones are minimum, so now we must move to the 3 and reduce to {10, 2,2,1}. The rightmost 2 can be reduced to give {10, 2,1,1,1}. There aren't enough aces in the deck to exchange for the remaining 2, and to do so would exceed the five-card limit. Required to keep a 10, there are no allowable reductions left. The algorithm has terminated.

Application of the two guidelines produced a methodical reduction to reveal all the possible combinations that add to fifteen AND include a 10. As we progress through the next stages of this problem, any time we feel the urge to include a 10-card in an attempt at a new entry, we can stop--with the confidence that "if it uses a 10, it has already been listed in stage one."

Stage two

Stage two: "How many different ways can you use five or fewer card-values to get a sum of 15--where the largest value is a 9? List them all." Apply the guidelines to construct the string of reductions: {9,6} {9,5,1} {9,4,2} {9,4,1,1} {9,3,3} {9,3,2,1} {9,3,1,1,1} {9,2,2,2} {9,2,2,1,1}. Required to keep a 9, there are no allowable reductions remaining. The algorithm has terminated.

Application of the two guidelines produced a methodical reduction to reveal all the possible combinations that add to fifteen AND include a 9. As we progress through the next stages of this problem, any time we feel the urge to include a 9-card in an attempt at a new entry, we can stop--with the confidence that "if it uses a 9, then it has already been listed in stage two." Not only can we be confident it has already been accounted for, but we know where to look--to make sure in the case of doubt.

Continue this process, removing one concern (card value) at each stage. It may seem a little trickier in later stages--especially as the five-card limit gets imposed. Eventually you get to the final stage of gathering preliminary data.

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Stage seven

Stage seven: “How many different ways can you use five or fewer card-values to get a sum of 15--where the largest value is a 4? List them all.” Apply the guidelines to construct the string of reductions: {4,4,4,3} {4,4,4,2,1} {4,4,3,3,1} {4,4,3,2,2} {4,3,3,3,2}. Required to keep a 4, add to 15 and use 5 or fewer elements, there are no allowable reductions remaining. The algorithm has terminated.

Termination

There are no possible solutions using only values three and under. Thus, the process of finding all possible numeric combinations, with the given constraints, is complete. Results from the seven stages above need only be compiled, and the number of ways a deck can be used to construct each needs to be computed (use the multiplication rule). Add those numbers to answer the original question.

Conclusion

This problem required a serious investigation into problem solving strategies, including two levels of the familiar “break the problem into smaller problems”. The solution’s design included a goal that the smaller problems yield disjoint solutions, so the final compilation would be a simple use of set union. Application of the sorting algorithm occurred at many levels, intertwining the four essential parameters of the problem. Arithmetic and counting techniques were practiced and drilled en masse. All this mathematics was motivated by a simple desire to become familiar with the ways to score ‘15 for 2’ in the card game called cribbage.

Using the "algorithm" described above, these are the possible combinations of numbers that add to 15, and the number of ways cards can so combine.

COMBINATION	# WAYS
{10,5}	16*4
{10,4,1}	16*4*4
{10,3,2}	16*4*4
{10,3,1,1}	16*4*6
{10,2,2,1}	16*6*4
{10,2,1,1,1}	16*4*4
term w/6	
{9,6}	4*4
{9,5,1}	4*4*4
{9,4,2}	4*4*4
{9,4,1,1}	4*4*6
{9,3,3}	4*6
{9,3,2,1}	4*4*4*4
{9,3,1,1,1}	4*4*4
{9,2,2,2}	4*4
{9,2,2,1,1}	4*6*6
term w/9	
{8,7}	4*4
{8,6,1}	4*4*4
{8,5,2}	4*4*4
{8,5,1,1}	4*4*6
{8,4,3}	4*4*4
{8,4,2,1}	4*4*4*4
{8,4,1,1,1}	4*4*4
{8,3,3,1}	4*6*4
{8,3,2,2}	4*4*6
{8,3,2,1,1}	4*4*4*6
{8,2,2,2,1}	4*4*4
term w/11	
{7,7,1}	6*4
{7,6,2}	4*4*4
{7,6,1,1}	4*4*6
{7,5,3}	4*4*4
{7,5,2,1}	4*4*4*4
{7,5,1,1,1}	4*4*4
{7,4,4}	4*6
{7,4,3,1}	4*4*4*4
{7,4,2,2}	*4*4*6
{7,4,2,1,1}	4*4*4*6
{7,3,3,2}	4*6*4

COMBINATION	# WAYS
{7,3,3,1,1}	4*6*6
{7,3,2,2,1}	4*4*6*4
{7,2,2,2,2}	4*1
term w/14	
{6,6,3}	6*4
{6,6,2,1}	6*4*4
{6,6,1,1,1}	6*4
{6,5,4}	4*4*4
{6,5,3,1}	4*4*4*4
{6,5,2,2}	4*4*6
{6,5,2,1,1}	4*4*4*6
{6,4,4,1}	4*6*4
{6,4,3,2}	4*4*4*4
{6,4,3,1,1}	4*4*4*6
{6,4,2,2,1}	4*4*6*4
{6,3,3,3}	4*4
{6,3,3,2,1}	4*6*4*4
{6,3,2,2,2}	4*4*4
term w/14	
{5,5,5}	4*1
{5,5,4,1}	6*4*4
{5,5,3,2}	6*4*4
{5,5,3,1,1}	6*4*6
{5,5,2,2,1}	6*6*4
{5,4,4,2}	4*6*4
{5,4,4,1,1}	4*6*6
{5,4,3,3}	4*4*6
{5,4,3,2,1}	4*4*4*4*4
{5,4,2,2,2}	4*4*4
{5,3,3,3,1}	4*4*4
{5,3,3,2,2}	4*6*6
term w/12	
{4,4,4,3}	4*4
{4,4,4,2,1}	4*4*4
{4,4,3,3,1}	6*6*4
{4,4,3,2,2}	6*4*6
{4,3,3,3,2}	4*4*4
term w/5	