They played by the fire
And under the stars …
The content was in the cards
And it still is.

An overview of elementary probability topics
with the standard deck of cards as the carrier
and the game of poker as the thread.

Presented by:

Robert N. Baker
Assistant Professor of Mathematics
UAS-Ketchikan

17 November 2001
at
American Mathematics Association of Two-Year Colleges
27th Annual Conference
Toronto, Canada

We humans do like to manipulate
things and information,
in work and play.

Given the logistics of living in a material world,

we tabulate information
about that which was manipulated.

Diverse people with common goals—trying to 'figure out' the math

can benefit greatly if we Cooperate
(Try it in the math classroom!)

The safety of a cooperative classroom, is the place

to teach and practice the art of civil competition

Games with math manipulatives--
--skill and drill with a tangible reason to pay attention to details.

Manipulation ...

of things to get at arithmetic

of numbers to get at algebra

of algebra to get at calculus

of things to get at discrete methods

of models to get at design

of measures to get at science

(and the “real” numbers)

is a lot of our life!

Manipulatives in the mathematics classroom for

Concrete experience
in a safe and collegial atmosphere, with

live investigations and concept development,
mathematical modeling, measurement,

remediations and the meaning of division ...

(time permitting?)

Cards in the classroom

The standard 52-card deck of cards is:

a mathematically non-trivial discrete set;

kinesthetically accessible;

visually and intellectually discernible;

cheap and easy to find;

refered to by many intro. texts in probability;

and has staying power and popularity
in our culture, and around the world;

with specials on “Random Variables”

“... which are neither random nor variable.”

A Socio/economic Model: Four “Suits”

CASTE:				
China a.d. 1000	“peasant”	military”	professional”	“religious”
Modern Surveys:				
“Circle one U.S., 1995	<$15,000/yr	<$45,000/yr.	<$100,000/yr.	>$100,000/yr

Representations (“pips”) used in the physical model (deck) of the theoretic model (society):

Chinese Pip-shape**:** a.d.1000	coins	strings of coins	tens of strings of coins	myriads of tens of strings of coins
Egyptian Pip-shape**:** a.d. 1200	sticks or wands	daggers or swords	precious stones or pentacles	chalices or mirrors
European	♣	♠	♦	♥
Pip-shape**:** a.d.1400	**“clubs”**	**“spades”**	**“diamonds”**	**“hearts”**

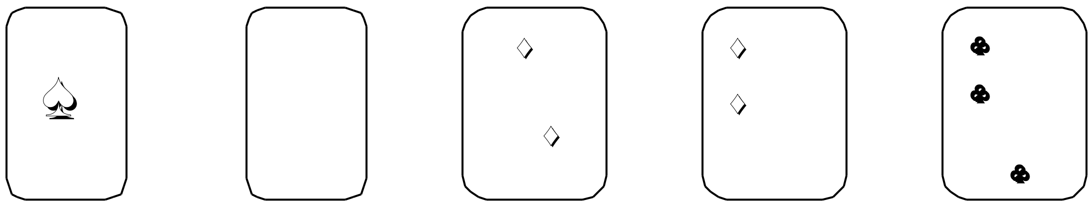
Further distinction relative to the model:

The most common social associations between castes are indicated by the dominant color of the card face, and of the pips.

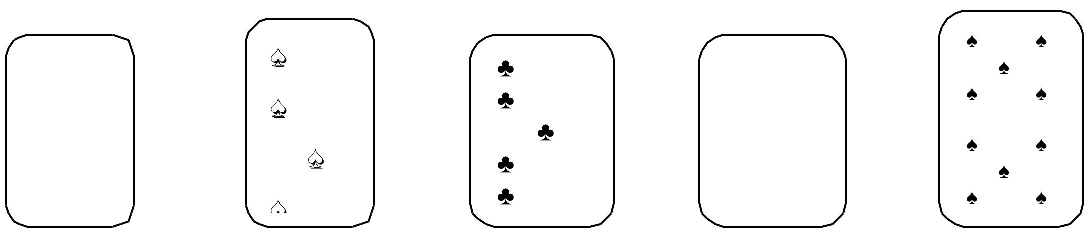
	black	**black**	**red**	**red**

Breakdown of castes/suits into 13 levels/kinds:

-ten of which are common: easily measurable/classifiable with numbers (1 through 10) used as category name:

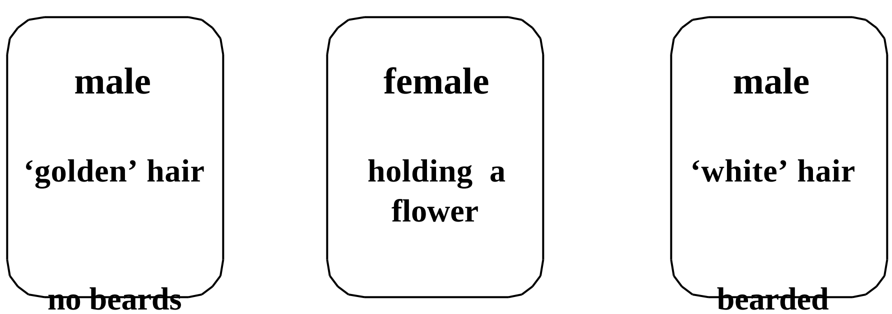


1 'Ace' 2 3 4 5



6 7 8 9 10

-three of which are less common: (the king, queen, and knave) not easily measured/ enumerated, represented only by their properties, where the symmetries are more intricate, and similarities between suits few:



knave 'jack' **queen** **king**

Partitions of the set (deck)

2 COLORS: black and red
26 of each

4 "SUITS": spades, clubs, diamonds, hearts
(respectively designated by pips shaped ♠, ♣, ♦, ♥)
13 in each suit

2 "TYPES": numeric and human patterns
('human' cards are also known as 'picture' or 'face' cards)
12 faces, 40 numbers

13 "KINDS": Ace, 2, 3, _ 10, Jack, Queen, King
('kinds' also known as 'denominations' or 'ranks' or 'levels')
4 in each kind

Classification through deduction

_ given the suit, can deduce the color
_ given the kind, can deduce the type

Note: $\nLeftarrow$ can't go the other way

INTELLIGIBLE as well as VISUAL

Notation desired, to write about the
individual elements of the universal set.

Scripted Notation: x_n , $x_{i,j}$

Prepare students for higher maths, in natural setting

1) Intro to reading subscripts
2) Use for non-computational information
   (esp. important for computer students)

"What is (most) important?"

If suits, use "suit sub kind": **d_7, s_a**

"the diamond seven" and "the spade ace"

If kinds, use "kind sub suit": **7_d , a_s**

"the seven of diamonds" and "the ace of spades"

Poker across continents and centuries--the what

<15th Century, Persian game *As ras*

15th Century,

Italian game *Il Frusso*, later *Primiera*
French game *LaPrime*, later *L'Amigu* or *L'Mesle*
English game *Post and Pair*
German game *Pochen*

1718: the game *Poque* in the Louisiana territory

1800: the game still dealt from a 20 card deck
("restricted domain" yields better hands!)

Full-deck Poker invented shortly thereafter
(average hand is lowered)

1860: Draw Poker introduced in America

1870: "Jacks or better" introduced, also known as "Jack-pots".

1999: Poker machines in every business in Montana,
Choose your game!

Encyclopedia Britannica, 1911

Poker--the how

1) Players are dealt one card at a time, from twice randomized 52-card deck, clockwise

2) until each has 5 cards.

Def'n: These 5 cards are called a player's "hand".

Def'n: A round of play is also called a "hand", as in "Each player is dealt one hand per hand."

2) To determine the winner of the round, players show and compare hands.

This game places "value" on easily recognized combinations of cards,

defined in terms of equivalence classes inherent in the standard deck.

First, we need to introduce:

Comparative-value
for the kinds

In poker, the 13 kinds
are assigned a (transitive) ordering.

<u>HIGH</u> <u>to</u> <u>LOW</u>

Aces, kings, queens, jacks, 10's, ... , 3's, 2's

A > k > q > j > 10 > 9 > ... > 3 > 2

This ordering is used in poker to:

1) determine the winner of a cut (see random decisions);

2) enable a concept/combination known as a "straight";

3) help well-define an order on "hands of the same type".

Types of **Hands**:
ie. valued combinations of 5 cards

Defined for easy identification with the mind and senses, a hand is considered valuable if the five cards include:

Type 1: based on the 13 "kinds"

Two of any one kind ("a pair" or 1P)

Two of any two kinds ("two pair" or 2P)

Three of any one kind ("three of a kind" or 3K)

Three of any one kind AND two of another kind
(called a "full house" or FH)

Four of any one kind ("four of a kind" or 4K)

Type II: based on the four "suits"

Five cards in any one suit ("a flush" or Fl.)

Types of Hands: valued combinations within the 5 card sets

Type Three: based on the "**ordering**"

as defined on the kinds (see above)

Five cards "in a row" ("a straight" or St.)

Note: In poker, an ‘ace’ may be used in a ‘straight’ either:
as a low card in the hand {A, 2, 3, 4, 5};
or as a high card in the hand {10, J, Q, K, A}.

Type 4: none of the above (“nota”)

Five cards not of the same suit,
not in consecutive order, and
no two of the same kind

Note: **A hand of renown is called a straight flush**--
five cards in a row AND all of the same suit.
Combining the natural and the imposed characteristics of the set (deck).

The highest of the straight flush hands,
thus the highest hand in the game of poker,
includes the 10, J, Q, K, Ace in one suit,
is called a “royal flush”.

Comparative value of the hands

In poker, the hands are given a (transitive) ordering.

HIGH to LOW
Str't. Flush, 4k, full house, flush, straight, 3k, 2 pr, 1 pr

SF > 4k > FH > Fl > St > 3k > 2pr > 1pr > nota

The winner of a round ("hand") is determined by a (physical) comparison of patterns in the cards in the hands.

Note: If no player has any of the above combinations, then that player with the highest card is the winner.

Example of nota:
{2, 4, 6, 8, A} beats {8, 10, J, Q, K}

In the case of a tie, ...

Comparative value of the hands

SF > 4k > FH > Fl > St > 3k > 2pr > 1pr > nota

If two players have a hand classified as (one of the above)

then, that one which includes the higher kind

is deemed the higher hand.

The order of the hands is nested on the order of the kinds!

(while the suits are all equal)

Comparative-value for the kinds

A > k > q > j > 10 > 9 > ... > 3 > 2

Comparative-value of the hands

SF > 4k > FH > Fl > St > 3k > 2pr > 1pr > nota

Special Cases:

Full-house: Compare for the highest kind in the triplet only.
Example: {A, A, A, 2, 2} beats {K, K, K, Q, Q}

Two-pair: Compare on each player's highest pair.
If that is a tie, then compare the lower pair.
If a tie still remains, then compare for highest fifth card
If a tie still remains, see "random decision"
Example: {2, 2, 3, 3, 6} beats {2, 2, 3, 3, 5}

Straight: Compare on each player's highest card.
If a tie, see "random decision"

Flush: Compare on each player's highest card.
If a tie, compare on second highest, then third, etc.
If identical kinds in different suits, see random decision

A pair: Compare for the highest pair.
If a tie, compare each player's highest other card.
If a tie, compare the second highest, then third if need.

RANDOM DECISIONS

Choose One: To determine a winner in a tie, effected players should agree to cut the deck for high (or low) card, flip a coin, draw straws, or etc. ... or "split the pot" ... or see "House Rules."

In poker, this ordering of hands is given as

"the rules of the game,"
circa 15th century.

It is not an arbitrary order!

The values
of two types of hand

compare inversely to the

probabilities of getting them
(dealt from a randomized deck)
circa 18th Century

Higher valued hands have lower probability of occurrence.

Probabilities can be computed

for five-card hands in straight poker
from a randomized, 52-card deck

by counting 5-card subsets of the deck
and using techniques from probability theory

Example: as a ratio,

Probability(you get a certain type of hand, say E) = Pr(E) =

$$= \frac{\text{the number of different E hands possible in the deck}}{\text{the number of different 5-card hands in the deck}}$$

Note:
The math involved in draw poker is a tad more involved, though leaves us with the same ordering for the hands.

First we get the denominator ...

… the number of “different” 5-card hands

(ie. the number of 5-element subsets of a 52-element set)
(ie. the number of ways to choose 5 items from 52)

By the multiplication rule for events, there are

52 * 51 * 50 * 49 * 48 = 311,875,200

ways to be dealt five cards from the deck.

But this computation implies a concern for the process
(in particular, the order in which the cards are dealt)
that is not reflected by evaluations in the game of poker.
This game values only the finished five-card product.

Each of the “different” five-card poker hands
can be dealt in
2 * 3 * 4 * 5 = 5! = 120
different ways.

Thus, there are

(311,875,200) / 120 = **2,598,960**

different five-card hands possible from a deck of 52 cards.

Activity:

Randomly assign groups of 4, through matching cards.

1) Individuals meet, find work space, introduce selves.

2) Assign duties. Each group will need someone to:

   a) Shuffle, arrange, keep track of tools (Logistics)

   b) Record data

   c) Technical support (calculation, data entry)

   d) Spokesperson

Activity:

1) Extract 5 cards from the deck;
for ease of discussion use an Ace, 2, 3, 4 and a 5.

2) Deal these cards face-up, one at a time, and construct
all the different ways that this hand can be dealt
(ie. the permutations of these 5 items).

3) Make a list of the 120 different orders in which these cards can be dealt (permutations of these 5 cards).

The “120”
different-ways-that-each-hand-can-be-dealt

Consider the following example:

Let: A poker hand include a 1, 2, 3, 4, and a 5.

Here is a list of the ways this hand can result,
listed “first-card-dealt on left to last-card-dealt on right”.

12345	13245	14235	15234
12354	13254	14253	15243
12435	13425	14325	15324
12453	13452	14352	15342
12534	13524	14523	15423
12543	13542	14532	15432
2 * 3	2 * 3	2 * 3	2 * 3

so far there are (2 * 3) * 4 ways to deal out this hand

This list includes only all of the ways to get this hand dealt where the ace is the first card received by the player.

We can generate four similar lists—with 2 the first card dealt, 3 first, the 4 and the 5 always first. Thus there are

$$(2 * 3 * 4) * 5 = 120$$

different ways to be dealt this one particular poker hand.

The same can be said for any particular poker hand.

Activity: The # of ________ hands.

(Start with the best, work down.)

I: For straight flush hands

a) List all of the different versions possible with cards from a standard deck.

b) Give the number of different versions possible from a standard deck.

c) Give the probability of being dealt such a hand in a game of straight poker from a randomized deck.

II: For 4-of-a-kind (4K) hands.

Repeat the above steps.

The number of straight flush hands

There are 4 possible suits from which to have 5 of,
to satisfy the flush part. How many straights in each?

Pick any one suit.

How many different 5-card straights can you make if just using the cards from that one suit?

Count them: {A, 2, 3, 4, 5} and {2, 3, 4, 5, 6} and ...

... {10, J, Q, K, A} makes 10.

There are 4 * 10 = 40 "different" straight-flush hands

The probability of getting a straight-flush hand

in straight poker,
five cards from a randomized 52-card deck, is

$$Pr(St.Fl.) = 40 / 2{,}598{,}960$$

$$\approx 0.000015 \approx 0$$

The Number of "four of a kind" hands

aka "a 4k hand"

There are 13 possible kinds from which to get four of.

There will be a fifth card in the hand.

Example: Given four aces,
the fifth card could be any of 48 cards left in the deck.
(Get out a deck and see!)
So there are 48 different ways to have four aces.

There are **13 * 48 = 624** "different" 4K poker hands

Thus, the probability of getting dealt a four-of-a-kind hand in straight poker, from a randomized 52-card deck, is

$$Pr(4k) = 624 / 2{,}598{,}960$$

$$\approx 0.00024 \approx 0$$

See handout for probabilities
for all of the different poker hands.

(five-card, straight poker
from a randomized deck)

==================================

What is the average poker hand?

"Discuss.

Not too heatedly."

Theoretic vs. experimental probabilities.

Activity: See tally worksheet.

Generate and compile data about 5 card poker.

1) Play as many hands as time allows ...

2) Compile data in two tables.

* For each round, document two distinct "outcomes":
i) the type of hands dealt to individual players
ii) the type of hand winning the round

3) Be prepared to read your frequencies aloud,so that class data can be compiled…to address:

"as n approaches infinity" ... ?

---

4) During one round, stop the dealing after everyone has been dealt exactly 4 cards. Each player should then consider two questions. Stop and discuss:

i) What is Pr (the 5th card will be helpful)?
ii) What is Pr (this will be the winning hand)?

NOTE:
These represent two "essentially different" concerns,
yet both settings are commonly described with the same "Conditional Probability" equations!

Handout for data gathering. “They played under the stars and by the fire ...”

Activity: Play as many hands of 5-card, straight poker as time allows. Gather data.

Instructions: Choose someone to deal first.
Randomize (shuffle). Randomize again (cut).
Player to dealer’s right cuts the cards; player to dealer’s left is dealt first, then clockwise.
Deal one card face down to each player. (Players may look at their own.)
Deal one card face up to each player. Repeat until each player has 5 cards.
Compare hands and determine the winner of the round.
Tally information for: hands that arose, winning hand, winning player.
Pass the deck to the left for the new dealer, and repeat above steps.

Tally for: Hands that were dealt.

Number of cards in deck ______

Total number of hands (5-card sets) dealt: __________

Tally for: Hands that won a round

Total number of rounds played: __________

For “Nota” hands that win a round, list winning high card here:

Machine Poker

1) Comparative values don’t work
in a one-player game.

2) As an American investment
the machine must make money

3) Corruption-control requires
that we keep track of
the **average payout** for each machine,

… a number which can be interpreted as

the average poker hand generated by that machine.

Which returns us to the question of ...

What is

the average poker hand?

An average is a number ...

so how do you get numbers out of hands?

Suggestion: Use a

Random Variable

(which is neither random nor variable!)

Machine Poker

Define/construct a random variable

(call it “payout”, with nickname “$”)

based on known probabilities, so that

if the laws of probability hold

(and watch your random-number generator)

then in the long run,

the costs to players will exceed the payout.

--choose numeric values for each type of hand,
do the computations; adjust until legal and profitable, etc.

(ex: a pair is 0, two pair is 1, ... , 4k is 200, St.Fl. is 400)

Activity:

What is the average poker hand?

see “Thelmo at the Boxford” worksheet

1) Generate a “probability distribution” for $
2) Compute the “expected value” for random variable $

$$E(\$) = \Sigma\, [(\$) * Pr(\$)]$$

3) What hand is assigned this number, E($), under the random variable $?

That hand is a reasonable candidate for “the average poker hand”

(for Thelmo’s machine)

Thelmo at the Boxford Cafe and Casino

Thelmo needed to know: "What is the average poker hand?" No one would really tell her. She decided that she could, as a start, find the average poker hand for one of the poker machines down at the Box'.

Thelmo set a goal of ten thousand plays. She wanted a large enough sample to assure reliable statistics, and an easy number to work with when computing those statistics.

Being a poor student, she bought a cup of water and watched others play--and kept a tally for 10,000 plays on the machine. The table below summarizes Thelmo's observations, and gives the machine's payoff rate. Each play cost one dollar, and payoffs were in dollars.

The categories	Outcome frequency	Payout
Straight Flush	1	100
4 of a kind	15	25
Full-house	90	8
Flush	130	5
Straight	250	5
3 of a kind	800	2
2 pairs	1500	1
Pair of Aces	514	1
Anything else	?	0

1. How many times did the machine give no payout? ____________

2. Define a Random Variable "$" to be the possible payouts on Thelmo's machine. Use Thelmo's data to complete the probability distribution for $.

$	100	25	8				
Pr($)							

3. Find: Pr($ ≤ 1) = __________ Pr($ ≥ 2) = __________

4. Compute the expected value for this machine: E($) = ___________

5. Give an interpretation of Ex($) to a potential player of machine poker.

6. What is the average hand for Thelmo's poker machine at the Box'?

Game Theory

Defn: In a **fair game**, the player's expected value and cost-to-play are equal.

Case 1: A fair game occurs when

payoff is inversely proportional to probability.

Examples:

If Pr(x) = 1/2, then a player with hand x should receive twice what she paid to play.

If Pr(x) = 1/10,000, then a player with hand x should receive 10,000 times what she paid to play.

Machine Gaming

In **one-player games**, such as on a machine,

the official and posted "payoff schedule"

(aka "random variable")

was NOT defined to yield a "fair game."

It WAS defined to

"honestly, legally, and fairly"

yield a predetermined percent profit

to the management, in the long run.

(given a good random-number generator
and honest employees.)

The Number of possible ways to get each kind of hand, from a standard deck of 52 randomized playing cards, in straight poker.

Each computation represents its own rationale for approaching the given counting problem. There are often several different ways to discover the one absolute answer to “how many ways can you ...?”

Here, “aCb” stands for the standard computation a!/[(a-b)! b!].

1 pair, no better i) 13 * 4C2 *12C3 *4^3
1,098,240
ii) (52 * 3 / 2!) (48 * 44 * 40 / 3!)

2 pairs, no better i) 13C2 * 4C2 * 4C2 * 11 * 4 **123,552**
ii) [(52 * 3 /2) (48 * 3 /2) / 2!] * 44
iii) [(13C1 * 4C2) (12C1 * 4C2) / 2!] * 11C1 * 4C1

3 of a kind, no better i) 13 * 4C3
* 12C2 * 4^2 **54,912**
ii) (52 * 3 * 2 / 3!) (48 * 44 /2!)
iii) 13 * 4C3 * (12 * 4C1 * 11 * 4C1 / 2!)

Straight, no better i) 10 * 4^5 - 40 **10,200**
(there are 40 “straight flush” hands!)

Flush, no better i) 13C5 * 4 - 40 **5,108**

Full-house i) 13 * 4C3 * 12 * 4C2 **3,744**
ii) (52 * 3 * 2 / 3!) (48 * 3 /2!)

4 of a kind i) 13 * 4C4 * 12 * 4C1 **624**
ii) (52 * 3 * 2 * 1 / 4!) * 48

Straight Flush i) 10 * 4 **40**

None of the above, “nota”: **1,302,540**
for these two methods, you need some of the above information
i) (hands without even a pair) - (hands that are flush or straight) 1,317,888 - 15,348 =
[(13 * 4) (12 * 4) (11 * 4) (10 * 4) (9 * 4) / 5!] - (10,200 + 5,108 + 40)

ii) (total number of hands) - (hands noted above as valued) 2,598,960 - 1,296,420 =
2,598,960 - (40+624+3744+5108+10,200+54,912+123,552+1,098,240)

Total number of different hands **2,598,960**

The Probabilities ...

... to get each type of hand in the game “straight poker” from a standard deck of 52 randomized playing cards.

These all use the formula: the probability of an event =

$$\frac{\text{(\# of elements in the event)}}{\text{(\# of elements in the sample space)}}$$

Let s = 2,598,960 (the number of different 5-card hands)

Type of hand	Method	Probability
Straight Flush	40/s	≈ .00001539
4 of a kind	624/s	≈ .00024010
Full house	3,744/s	≈ .00144058
Flush	5,108/s	≈ .00196540
Straight	10,200/s	≈ .00392464
3 of a kind	54,912/s	≈ .02112845
two pairs	123,552/s	≈ .04753902
one pair	1,098,240/s	≈ .42256903
none of the above	1,302,540/s	≈ .50117739