

Re-Teaching Algorithms: Who is More Bored, You or the Students?

Laura Bracken

Lewis-Clark State College

bracken@lsc.edu

1. Introduction: Why are we here?
2. Sample misunderstandings
3. One Strategy for Re-Teaching:
4. Requirements:
You must be willing to do:
 - *Some student training in group work and social interaction.*
 - *Some community building.*
 - *Some shifting of responsibility for all explaining from the instructor to the student.*
 - *Assessments that motivate the student to engage in learning.*
5. Practical Advice
 - *Sandwich activities between your whole-class introduction and summary.*
 - *Grade work as a group using a rubric. Substitute for attendance/participation grades. Ask for verbal as well as written explanations. Look at everyone's work some of the time.*
 - *Don't answer too many questions; ask more questions than you answer.*
 - *Emphasize that you are in control and that it is not a race. Have an ending activity in your class so that fast work is not overly rewarded.*
 - *Get to know students personally as you move from group to group, even if it takes the group off task occasionally.*
6. Sample Activities
*(spacing in these handouts does not provide sufficient room for student work)
7. Writing these activities.
 - *Remind or ask students for prerequisite knowledge. May reinforce a pre-activity lecture.*
 - *Limit what is taught in each example. Use more examples rather than complicated examples.*
 - ***Require students to explain concepts and algorithms in their own words.***
 - *Include sufficient but not overwhelming number of checkpoints.*
 - *Provide sufficient practice to allow for variation in group speed.*
 - *Occasionally use group reporting to learn from others' work.*

INVESTIGATION

Multiples of Ten

The natural numbers are $\{1, 2, 3, 4, \dots\}$. We call numbers like 10, 100, 1,000 and 10,000 *multiples of ten* since each is the product of 10 and a natural number.

1. **Count the number of zeros in the factors and the number of zeros in the product.**

| | Total zeros in factors | Zeros in product |
|--------------------------|------------------------|------------------|
| $(10)(43) = 430$ | | |
| $(100)(43) = 4,300$ | | |
| $(1,000)(43) = 43,000$ | | |
| $(10,000)(43) = 430,000$ | | |

2. **Describe how to find the product of a whole number and a multiple of ten without doing regular multiplication.**

Use this process to quickly find each product.

3. $1,000(21)$ 4. $(100)(400)$ 5. $10,000(4)$
6. $10 \cdot 88$ 7. $75 \cdot 10$ 8. $(100)(442)$

We can use a similar process to find the product of numbers that have only one digit that is not zero. For example, both 20,000 and 400 have only one *non-zero digit*.

$$(20)(40) = 800$$

$$(200)(40,000) = 8,000,000$$

$$(60)(5) = 300$$

$$(600)(5,000) = 3,000,000$$

9. **Describe how to multiply whole numbers that each only have one non-zero digit.**

Quickly find each product.

10. $20 \cdot 900$ 11. $2(9,000)$ 12. $200 \cdot 9,000$
13. $(300)(40)$ 14. $50 \cdot 60,000$ 15. $80 \cdot 8$

Answers. 3. 21,000; 4. 40,000; 5. 40,000; 6. 880; 7. 750; 8. 44,200; 10. 18,000; 11. 18,000; 12. 1,800,000; 13. 12,000; 14. 3,000,000; 15. 640.

Instructor Check.

Investigation: Simplifying Exponential Expressions

We can rewrite an exponential expression as a product. For example, $x^3 = x \cdot x \cdot x$.

1. Rewrite x^7 as a product.

When we multiply two exponential expressions with the same base, we can rewrite both expressions as products. For example, $x^3 \cdot x^7 = (x \cdot x \cdot x)(x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x)$. Since x is multiplied by itself 10 times, $x^3 \cdot x^7 = x^{10}$.

2. Multiply $x^9 \cdot x^5$.

The product rule of exponents describes how to multiply any two exponential expressions with the same base: $x^m x^n = x^{m+n}$.

3. Describe the product rule of exponents in words.

4. Explain why this is called the *product* rule of exponents.

Simplify each expression using the product rule of exponents.

5. $x^4 x^5$

6. $x \cdot x^5$

7. $x^3 x^5 x^5$

Answers. 5. x^9 ; 6. x^6 ; 7. x^{13} .

Instructor Check.

We can divide exponential expressions with the same base using a similar method.

8. Rewrite the exponential expressions in the numerator and denominator of $\frac{x^7}{x^3}$ as products.

When the numerator and the denominator of a fraction are the same, the fraction is equal to 1. For example, $\frac{x}{x} = 1$. So, $\frac{x^7}{x^3} = \frac{x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x}{x \cdot x \cdot x} = 1 \cdot 1 \cdot 1 \cdot x^4 = x^4$.

9. Rewrite the exponential expressions in the numerator and denominator of $\frac{x^8}{x^6}$ as products.

10. Replace any fraction that has the same numerator and denominator with 1.

11. Simplify the expression in problem 10.

The quotient rule of exponents describes how to divide any two exponential expressions with the same base: $\frac{x^m}{x^n} = x^{m-n}$.

12. Describe the quotient rule of exponents in words.

13. Explain why this is called the *quotient* rule of exponents.

Simplify each expression.

14. $\frac{x^8}{x^3}$

15. $\frac{x^8}{x}$

16. $\frac{x^8 x^2}{x^3}$

Answers. 14. x^5 ; 15. x^7 ; 16. x^7 .

Instructor Check.

Investigation: Fractions, Rational Expressions, and Multiplication

1. Simplify $(6xy^2)(7xy^5)$.

2. Simplify $\frac{40x^2y^{10}}{16xy^3}$.

Answers. 1. $42x^2y^7$; 2. $\frac{5xy^7}{2}$.

3. Describe what is happening in each step of the multiplication problem.

$$\begin{aligned} & \frac{5}{8} \cdot \frac{2}{15} \\ &= \frac{5 \cdot 2}{8 \cdot 15} && \text{Step 1.} \\ &= \frac{\overset{1}{\cancel{5}} \cdot \overset{2}{\cancel{2}}}{\underset{4}{\cancel{8}} \cdot \overset{1}{\cancel{15}} \cdot 3} && \text{Step 2.} \\ &= \frac{1 \cdot 1}{4 \cdot 3} && \text{Step 3.} \\ &= \frac{1}{12} && \text{Step 4.} \end{aligned}$$

4. Multiply $\frac{15}{32} \cdot \frac{8}{9}$. Show your thinking and work.

5. Explain how to multiply fractions. You may write in steps or in a paragraph.

Instructor Check

6. Describe what is happening in each step of the multiplication problem.

$$\begin{aligned} & \frac{9x}{10y} \cdot \frac{2y^2}{15x^3} \\ &= \frac{9x \cdot 2y^2}{10y \cdot 15x^3} && \text{Step 1.} \\ &= \frac{\overset{3}{\cancel{9}}x \cdot \overset{1}{\cancel{2}}y^2}{\underset{5}{\cancel{10}}y \cdot \overset{1}{\cancel{15}}x^3} && \text{Step 2.} \\ &= \frac{3xy^2}{25x^3y} && \text{Step 3.} \\ &= \frac{3y}{25x^2} && \text{Step 4.} \end{aligned}$$

7. **Multiply** $\frac{21x^2}{40y^5} \cdot \frac{16y^2}{63x^8}$. **Show your thinking and work.**

8. **Describe what is happening in each step of the multiplication problem.**

$$\frac{9(x-1)(x+4)}{10x(x+6)} \cdot \frac{x(x+6)}{15(x-1)}$$

$$= \frac{9(x-1)(x+4) \cdot x(x+6)}{10x(x+6) \cdot 15(x-1)} \quad \text{Step 1.}$$

$$= \frac{\overset{3}{\cancel{9}}(\overset{1}{\cancel{x-1}})(x+4) \cdot \overset{1}{\cancel{x}}(\overset{1}{\cancel{x+6}})}{10\overset{1}{\cancel{x}}(\overset{1}{\cancel{x+6}}) \cdot \overset{5}{\cancel{15}}(\overset{1}{\cancel{x-1}})} \quad \text{Step 2.}$$

$$= \frac{3(x+4)}{50} \quad \text{Step 3.}$$

9. **Multiply** $\frac{2(x-2)(x+1)}{3x(x+4)} \cdot \frac{6x^2(x-4)}{9(x+1)(x-2)}$. **Show your thinking and work.**

10. **Describe what is happening in each step of the multiplication problem.**

$$\frac{9(x^2+3x-4)}{10x^2+60x} \cdot \frac{x^2+6x}{15x-15}$$

$$= \frac{9(x-1)(x+4)}{10x(x+6)} \cdot \frac{x(x+6)}{15(x-1)} \quad \text{Step 1.}$$

$$= \frac{9(x-1)(x+4) \cdot x(x+6)}{10x(x+6) \cdot 15(x-1)} \quad \text{Step 2.}$$

$$= \frac{\overset{3}{\cancel{9}}(\overset{1}{\cancel{x-1}})(x+4) \cdot \overset{1}{\cancel{x}}(\overset{1}{\cancel{x+6}})}{10\overset{1}{\cancel{x}}(\overset{1}{\cancel{x+6}}) \cdot \overset{5}{\cancel{15}}(\overset{1}{\cancel{x-1}})} \quad \text{Step 3.}$$

$$= \frac{3(x+4)}{50} \quad \text{Step 4.}$$

11. **Multiply** $\frac{6(x^2-1)}{(x^2+4x+5)} \cdot \frac{(x^2+8x+16)}{3(x^2+4x+3)}$. **Show your thinking and work.**

12. **Explain how to multiply rational expressions. Be complete.**

Instructor check.

For common factors to simplify to 1, there must be only *multiplication in the numerator and the denominator*. For example, we know $\frac{5 \cdot 3}{5 \cdot 8} = \frac{\cancel{5} \cdot 3}{\cancel{5} \cdot 8} = \frac{3}{8}$. When the operation changes to *addition*, numbers that are the same in the numerator and denominator do *not* simplify to 1: $\frac{\cancel{5} + 3}{\cancel{5} + 8} \neq \frac{3}{8}$.

13. Describe the mistake made in completing the following problem.

$$\begin{aligned}
 & \frac{x^2 + 5x + 6}{x^2 - 16} \cdot \frac{x^2 - 4x}{x^2 + 3x + 2} \\
 &= \frac{\cancel{1}x^2 + 5x + 6}{x^2 - 16} \cdot \frac{\cancel{1}x^2 - 4x\cancel{1}}{x^2 + 3x + 2} \\
 &= \frac{5x + 6 \cdot (-4)}{(-16)(3 + 2)} \\
 &= \frac{5x + 6 \cdot (-4)}{(-16)(5)} \\
 &= \frac{5x + 6}{20}
 \end{aligned}$$

14. Redo the problem correctly.

Instructor Check

Investigation: Common Denominators

1. Describe what is happening in each step of the addition problem.

$$\begin{aligned} & \frac{3}{10} + \frac{4}{15} \\ &= \frac{3}{10} \cdot \frac{3}{3} + \frac{4}{15} \cdot \frac{2}{2} && \text{Step 1.} \\ &= \frac{9}{30} + \frac{8}{30} && \text{Step 2.} \\ &= \frac{17}{30} && \text{Step 3.} \end{aligned}$$

2. A common denominator is not necessary to multiply or divide fractions. Why is a common denominator necessary for addition and subtraction of fractions?
3. Explain how to find a common denominator of two fractions.

Instructor Check.

4. In the problem $\frac{3}{10} + \frac{4}{15}$, we could have used 150 as the common denominator. Add these fractions using this common denominator instead of 30. Show all steps.
5. Describe any advantages of using the smallest possible common denominator (also called the *lowest common denominator*) in adding two fractions.
6. Add $\frac{2}{175} + \frac{9}{35}$. The final sum must be in lowest terms.

Addition of rational expressions also requires a common denominator. We usually want to use the lowest common denominator, also called the LCD.

Example 1. The lowest common denominator (LCD) of $\frac{1}{x+4}$ and $\frac{1}{2x-3}$ is $(x+4)(2x-3)$.

Example 2. The lowest common denominator of $\frac{1}{9(x+4)}$ and $\frac{1}{12(2x-3)}$ is $36(x+4)(2x-3)$.

Example 3. The LCD of $\frac{1}{x(x+4)(x-2)}$ and $\frac{1}{x(x+6)(x-2)}$ is $x(x+4)(x+6)(x-2)$.

7. Based on these examples, explain how to determine the lowest common denominator of two rational expressions.

Instructor Check.

Complete the table.

| | Rational Expression | Rational Expression | Least Common Denominator |
|-----|------------------------|--------------------------|--------------------------|
| 8. | $\frac{1}{3x+7}$ | $\frac{1}{x+1}$ | |
| 9. | $\frac{1}{4(x-10)}$ | $\frac{1}{18(x-2)}$ | |
| 10. | $\frac{1}{4x}$ | $\frac{1}{18(x-2)}$ | |
| 11. | $\frac{1}{4x}$ | $\frac{1}{18x(x-2)}$ | |
| 12. | $\frac{1}{(x-3)(x+2)}$ | $\frac{1}{12(x-3)(x+2)}$ | |
| 13. | $\frac{1}{10x^2(x+4)}$ | $\frac{1}{7x}$ | |
| 14. | $\frac{1}{x-7}$ | $\frac{1}{x}$ | |

Answers. 8. $(3x+7)(x+1)$; 9. $36(x-10)(x-2)$; 10. $36x(x-2)$; 11. $36x(x-2)$; 12. $12(x-3)(x+2)$; 13. $70x^2(x+4)$; 14. $x(x-7)$.

15. Describe the mistake made in completing the following problem. Then, do the problem correctly.

Problem: Find the lowest common denominator of $\frac{1}{12(x-3)(x-2)}$ and $\frac{1}{12(x+3)(x-2)}$.

Answer: $12(x-3)(x+3)(x-2)(x-2)$.

16. Describe the mistake made in completing the following problem. Then, do the problem correctly.

Problem: Find the lowest common denominator of $\frac{1}{12x}$ and $\frac{1}{12x^3}$.

Answer: $12x^4$.

17. Describe the mistake made in completing the following problem. Then, do the problem correctly.

Problem: Find the lowest common denominator of $\frac{1}{6(x-4)}$ and $\frac{1}{9(x+8)}$.

Answer: $3(x-4)(x+8)$.

Instructor check.

Investigation: Equations and Solutions

An equation is a statement that two expressions are equal. Equations may be always true, sometimes true, or always false. If an equation is always true, it is called an *identity*. An equation that is never true is called an *inconsistent equation*.

1. Classify each equation as an identity or as inconsistent.

A. $-1 + 4 = 3$

B. A meter is the same length as a yard.

C. $5 + 7 = 10$

D. $6 \div 3 = 2$

E. $79\% = 0.79$

F. 1 dime = 25 cents

When an equation includes a variable, the truth of the equation may depend on the value of the variable. Think about the equation $x + 1 = 10$. If the value of x is 9, this equation is true. If the value of x is any other number, the equation is not true. When an equation is only sometimes true, it is called a **conditional equation**.

Replace the variable with the given value, evaluate, and determine if the equation is true or false.

2. $x + 7 = 13$ when $x = 4$

3. $x + 7 = 13$ when $x = 6$

4. $x + 9 = 10$ when $x = -1$

5. $x + 20 = 2x + 4$ when $x = 8$

6. $x + 3 = 3 + x$ when $x = 10$

7. $5(x - 1) = 4(x + 3) - 14$ when $x = 3$

Answers. 1. Identities: A, D, E. Inconsistent: B, C, F; 2. False; 3. True; 4. False; 4. False; 5. False; 6. True; 7. False.

An equation that includes a variable is not always a conditional equation. For example, we found that $x + 3 = 3 + x$ is true when $x = 10$. However, it is also true when $x = 6$ or when $x = 1,000,000$ or when x is any real number. Since it is always true for any value, $x + 3 = 3 + x$ is an **identity**.

10. Explain why $x + 3 = 3 + x$ is always true.

11. Write another example of an equation with a variable that is an identity.

Instructor Check.

An equation that includes a variable may also be always false. Think about the equation $x + 3 = x + 5$. The sum of a number and 3 can never be equal to the sum of the same number and 5. So, there is no value for the variable that can make this equation true. This equation is said to be **inconsistent**.

12. Is $2(x + 4) = 2x + 8$ an identity, a conditional equation, or an inconsistent equation? Explain.

13. Is $2(x + 4) = 2x + 9$ an identity, a conditional equation, or an inconsistent equation? Explain.

14. Write another equation that is inconsistent.

15. Why do you think the word **inconsistent** is used to describe equations that are always false?

16. Why do you think the word **identity** is used to describe equations that are always true?

17. Why do you think the word **conditional** is used to describe equations that are sometimes true?

Instructor Check

Investigation: Rearranging Equations

When an equation is “solved for a variable”, it is rearranged so that the variable is isolated on one side of the equation. For example, the formula $P = 2L + 2W$ is “solved for P .” Because variables remain on both sides of the equation, the rearranged equation is not “simpler.” However, it is often more useful.

1. **Identify the properties of equality used in the example to solve the equation $P = 2L + 2W$ for W . (This is the formula for the perimeter of a rectangle.)**

| | | |
|---------|---------------------------------------------|--------------------------------------------------|
| Step 1. | $P = 2L + 2W$ $P - 2L = 2L + 2W - 2L$ | <i>Property of Equality used:</i> _____ |
| Step 2. | $P - 2L = 2W$ | <i>Combine like terms.</i> |
| Step 3. | $\frac{P}{2} - \frac{2L}{2} = \frac{2W}{2}$ | <i>Property of Equality used:</i> _____ |
| Step 4. | $\frac{P}{2} - L = W$ | <i>Simplify.</i> $\frac{2L}{2} = L$ |
| Step 5. | $W = \frac{P}{2} - L$ | <i>Reverse the sides (property of symmetry).</i> |

2. **Use this equation to find the width of a rectangle if the perimeter is 15 feet and the length is 4 feet.**

3. **Identify the properties of equality used in the example to solve the equation $A = \frac{1}{2}bh$ for h . (This is the formula for the area of a triangle.)**

| | | |
|---------|------------------------------------|--------------------------------------------------|
| | $A = \frac{1}{2}bh$ | |
| Step 1. | $2A = 2\left(\frac{1}{2}bh\right)$ | <i>Property of Equality used:</i> _____ |
| Step 2. | $2A = bh$ | <i>Simplify.</i> $2 \cdot \frac{1}{2} = 1$ |
| Step 3. | $\frac{2A}{b} = \frac{bh}{b}$ | <i>Property of Equality used:</i> _____ |
| Step 4. | $\frac{2A}{b} = h$ | <i>Simplify.</i> $\frac{b}{b} = 1$ |
| Step 5. | $h = \frac{2A}{b}$ | <i>Reverse the sides (property of symmetry).</i> |

4. **Use this equation to find the height of a rectangle if the area is 6 square feet and the base is 2 feet.**

5. **To rearrange the formula $V = LWH$ for W , will the addition property or the subtraction property of equality need to be used? Explain.**

Instructor Check

Rearrange each equation by solving for the specified variable. Show all steps.

6. Solve for y : $3x + 2y = 12$

7. Solve for y : $-6x + 2y = 30$

8. Solve for h : $E = mgh$

9. Solve for h : $A = \frac{1}{2}(b_1 + b_2)h$

10. Solve for y : $12x + 15y = 30$

Answers: 6. $y = -\frac{3}{2}x + 6$; 7. $y = 3x + 15$; 8. $h = \frac{E}{mg}$; 9. $h = \frac{2A}{(b_1 + b_2)}$; 10. $y = -\frac{4}{5}x + 2$.

11. Explain how to rearrange an equation to solve for a particular variable.

Instructor Check.

| Names | | POINTS |
|-------|----------------------------------------------|--------|
| | Descriptions (2 points). | |
| | Describe the order of operations (3 points). | |
| | Evaluate 18 - 25 (3 points). | |
| | Evaluate 26 - 37 (2 points). | |
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| | Evaluate 26 - 37 (2 points). | |