HOW LONG YOU CAN SURVIVE DEPENDS ON HOW WELL CALCULATED YOU DRINK

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Abstract

The function $l(n) = n\ln\left(\frac{1}{E_0} - \frac{1-E_0}{E_0}e^{-\frac{B}{n}}\right)$ is introduced through an interesting model. The maximum of $l(n)$ is then obtained. Several standard calculus techniques are employed to find the solution. The problem solving process is challenging and the answer is surprising. The paper also offers a glimpse of the use of mathematical symbolic manipulation utility. Finally, a possible generalization is given as a conjecture.

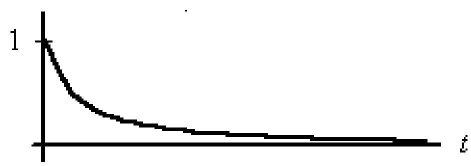
1. THE STORY BEGINS

Robinson C. is stranded on an island with *B* cans of beer and a handgun with a single bullet. There is no any other source of food on the island. Robinson figures that in order to increase his chances of being rescued, he has to keep himself alive for as long as possible and in the mean time, keep himself strong enough so that he can fire his gun for help if he spots a ship or a plane. How should he drink the beer in order to achieve this goal?

2. THE FUNCTIONS

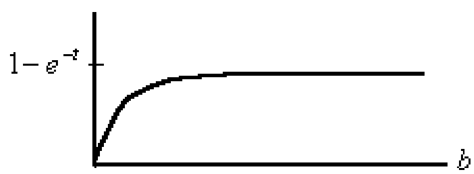
Let $t > 0$ denote time in days. Suppose Robinson's energy level is given by the function

(2.1) $$E(t) = e^{-t}$$



if he does not have any intake of food. If he drinks *b* cans of beer at time *t*, his energy level will have the following instantaneous improvement

(2.2) $$\Delta E(t,b) = \left(1 - e^{-t}\right)\left(1 - e^{-b}\right)$$



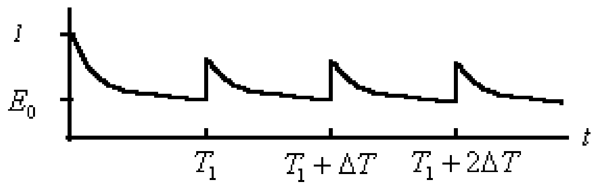
Now let $\hat{E}(t)$ denote the actual energy level at time *t*, and let E_0 denote the minimum energy level Robinson has to maintain. He wants to keep $\hat{E}(t) \geq E_0$ for as long as possible.

Robinson decides to take his first drink of $b = \frac{B}{n}$ cans of beer at time $T_1 = -\ln E_0$, that is when his energy level drops down to E_0 for the first time. After that, he will drink $b = \frac{B}{n}$ cans of beer every time his energy level reaches E_0, that is at each of the moments $T_1 + \Delta T$, $T_1 + 2\Delta T$, …, $T_1 + (n-1)\Delta T$. He has to find the optimal *n* and ΔT so that $\hat{E}(t) \geq E_0$ for the maximal length of time.

3. THE SOLUTION

At time $T_1 = -\ln E_0$,

$$\text{(3.1)} \qquad \begin{aligned} \hat{E}(T_1) &= e^{-T_1} + \Delta E\left(T_1, \tfrac{B}{n}\right) \\ &= E_0 + \left(1 - E_0\right)\left(1 - e^{-\frac{B}{n}}\right) \end{aligned}$$



The time it takes for $\hat{E}(T_1)$ to drop to E_0 is

$$\text{(3.2)} \qquad \begin{aligned} \Delta T &= T_1 - \left[-\ln \hat{E}(T_1)\right] \\ &= -\ln E_0 + \ln\left[E_0 + \left(1 - E_0\right)\left(1 - e^{-\frac{B}{n}}\right)\right] \\ &= \ln\left(\frac{1}{E_0} - \frac{1 - E_0}{E_0} e^{-\frac{B}{n}}\right) \end{aligned}$$

therefore the total number of days Robinson can survive is

$$\text{(3.3)} \qquad \begin{aligned} L(n) &= T_1 + n\Delta T \\ &= -\ln E_0 + n \ln\left(\frac{1}{E_0} - \frac{1 - E_0}{E_0} e^{-\frac{B}{n}}\right) \end{aligned}$$

Now we need only to find the maximum of

$$\text{(3.4)} \qquad l(n) = n\Delta T = n \ln\left(\frac{1}{E_0} - \frac{1 - E_0}{E_0} e^{-\frac{B}{n}}\right)$$

By choosing some specific values of E_0 and B, we can graph the function $l(n)$ on a graphing calculator. From these graphs, we may speculate that $l(n)$ is an increasing function and that $\lim_{n \to \infty} l(n)$ exists, i.e., it has a horizontal asymptote. This is indeed the case, as we will prove in Section 4. We now proceed to find the limit $\lim_{n \to \infty} l(n)$ since the graphs we obtained suggest that the limit is the maximum of $l(n)$.

Let $x = 1/n$ and apply L'Hopital's rule, we find

$$\lim_{n\to\infty} l(n) = \lim_{x\to 0^+} \frac{\ln\left(\frac{1}{E_0} - \frac{1-E_0}{E_0} e^{-Bx}\right)}{x} \tag{3.5}$$
$$= \lim_{x\to 0^+} \frac{\frac{1-E_0}{E_0} Be^{-Bx}}{\frac{1}{E_0} - \frac{1-E_0}{E_0} e^{-Bx}}$$
$$= \frac{1-E_0}{E_0} B$$

Thus, the maximum number of days Robinson can keep himself alive and well is

$$L_{\max} = -\ln E_0 + \frac{1-E_0}{E_0} B \tag{3.6}$$

and this can be achieved by drinking continuously at the rate of

$$R = \frac{B}{\frac{1-E_0}{E_0} B} = \frac{E_0}{1-E_0} \tag{3.7}$$

cans of beer per day, starting at the moment $T_1 = -\ln E_0$. For example, suppose Robinson has twelve (12) cans of beer and he needs to keep $\hat{E}(t) \geq E_0 = 0.5$, then he can survive a maximum of $L_{\max} = -\ln 0.5 + \frac{1-0.5}{0.5} \times 12 \approx 12.69$ days and he has to drink continuously at the rate $R = \frac{0.5}{1-0.5} = 1$ can of beer per day starting at the $T_1 = -\ln 0.5 \approx 0.69^{\text{th}}$ day.

4. DISCUSSION

4.1. Is $l(n)$ really increasing?

First we point out that $l(n)$ is differentiable. To show that $l'(n) > 0$, we will show that $l'(0) > 0$, $l'(\infty) \geq 0$, and that $l''(n) < 0$ (i.e., $l'(n)$ is decreasing). It is relatively easy to find that

$$l'(n) = \ln\left[\frac{1-(1-E_0)e^{-\frac{B}{n}}}{E_0}\right] - \frac{B(1-E_0)e^{-\frac{B}{n}}}{n\left[1-(1-E_0)e^{-\frac{B}{n}}\right]} \tag{4.1}$$

Note that $l'(n)$ is again differentiable, so we differentiate it one more time and find

(4.2) $$l''(n) = -\frac{B^2(1-E_0)e^{-\frac{B}{n}}}{n^3\left[1-(1-E_0)e^{-\frac{B}{n}}\right]^2} < 0$$

This shows that $l'(n)$ is decreasing. It can be verified that

(4.3) $$\lim_{n\to 0^+} l'(n) = -\ln E_0 > 0$$

and that

(4.4) $$\lim_{n\to\infty} l'(n) = 0$$

(4.2), (4.3) and (4.4) together implies that $l'(n) > 0$, i.e., $l(n)$ is an increasing function.

4.2. What can technology do?

A good calculator or computer software can reduce human error and free us from tedious computations. For instance, our computations in (4.1) and (4.2) are verified by Mathcad. The limits (3.5), (4.3) and (4.4) can also be found by using reliable algebraic software or calculators. However, no technology can perform creative thinking, and no software can set up mathematical models for us.

5. THE HAPPY ENDING

All right, we are all very concerned about Robinson's situation. Well, he drinks his 12 cans of beer very carefully. 300 hours after landing on the island, Robinson sees a flying object. He fires his gun, gets rescued by Peter Pan, and teaches calculus happily thereafter.

6. PROBLEMS AND QUESTIONS

1) What characteristics does the function $E(t)$ in (2.1) possess? (Answer: $E(0)$ is a positive number; $E(t)$ is decreasing; $E(t)$ approaches zero as t approaches infinity.) Can we go through the whole computation again with another function that has the same characteristics?

2) What characteristics does the function $\Delta E(t,b)$ in (2.2) possess? (Answer: $\Delta E(t,b)$ depends on $(1-e^{-t})$ – the difference between the highest energy level $E(0)=1$ and the current energy level $E(t)=e^{-t}$; $\Delta E(t,0)=0$; and as $b \to \infty$, $\Delta E(t,b) \to (1-e^{-t})$ – the largest possible energy improvement.) Can we go through the whole computation again with another function that has the same characteristics?

3) Verify (3.1) and (3.2).

4) Verify (4.1) through (4.4).

5) The conventional way to prove a differentiable function being increasing is to show its first derivative being positive. Why didn't we use this method to prove $l'(n)>0$ in Section 4? (Answer: it is hard to verify $l'(n)>0$ from (4.1).) Why is that (4.2), (4.3) and (4.4) together implies $l'(n)>0$? (Hint: Sketch a function with these properties.) To reach our conclusion, what basic property must $l'(n)$ possess? (Answer: $l'(n)$ must be continuous.) Can you think of other ways to prove a function being increasing if it is difficult to verify its first derivative being positive?

6) It is difficult for a person to drink "continuously." So from a practical point of view, what is the second best thing Robinson can do? (Hint: $l'(n)>0$. Answer: Drink as frequently as possible.)

7) Suppose $B=12$. How long can Robinson keep his energy level above 0.5 if he decides to keep it above 0.8 until all beer is consumed and then let his energy level drop? (Answer: about 3.69 days.)

8) You may come up with other ways to consume the beer. How do we prove (or disprove) that our solution is indeed the best?

9) **A conjecture.** Our result will still hold if
   a) $E(t)$ is decreasing and convex with $\lim_{t\to\infty} E(t)=$constant; and
   b) $\Delta E(t,b)$ is increasing and concave in *b* with $\lim_{b\to\infty} \Delta E(t,b)=1-E(t)$.

   More specifically, under these two conditions, the maximum length of time we can keep $\hat{E}(t) \ge E_0$ is $L_{\max} = T_1 + \lim_{n\to\infty} n\left(T_1 - E^{-1}\left(E(T_1) + \Delta E\left(T_1, \frac{B}{n}\right)\right)\right)$, where $T_1 = E^{-1}(E_0)$.