Investigating Involutions of the Plane

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The elementary algebra functions f, g, h and e, defined by $f(x) = -x$, $g(x) = \frac{1}{x}$, $h(x) = \frac{-1}{x}$ and $e(x) = x$ share the property of being their own inverses. Such functions are often called involutions. In this paper we investigate involutions of the plane, restricting our attention to linear transformations which fix the origin so that we can employ matrices.

What are some examples? Reflections across lines through the origin immediately come to mind. A rotation of 180° about the origin is another. Are these the only ones? It turns out that there are many others. We take an analytic approach to classify the linear involutions of the plane which fix the origin. Dropping the restriction that the origin be fixed adds nothing to the types we will discover - it just moves them around, such as with reflections across lines not through the origin.

A linear transformation T which fixes the origin is given by the equations

$$x' = ax + by$$

$$y' = cx + dy, \text{ where } ad - bc \neq 0$$

The condition $ad - bc \neq 0$ guarantees that the transformation is invertible; $ad - bc$, of course, is the determinant of A, the coefficient matrix for T:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

T will be its own inverse if and only if $A^2 = I$, the identity matrix; that is,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Now $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2 + bc & ab + bd \\ ca + dc & cb + d^2 \end{bmatrix}$.

We thus get four conditions which must be satisfied by the entries of A:

(i) $a^2 + bc = 1$

(ii) $ab + bd = 0$

(iii) $ca + dc = 0$

(iv) $cb + d^2 = 1$

We now examine four cases:

Case (1) $b = c = 0$. Then $a^2 = d^2 = 1$, yielding the four matrices

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad R_{180} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$R_x = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad R_y = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

These represent the identity transformation, the rotation about the origin through 180° (Figure 1a), the reflection across the x-axis (Figure 1b), and the reflection across the y-axis, respectively. Other reflections will appear as special cases of case (4) below.

Note: All figures herein were generated using a Texas Instruments TI-92 calculator program. The program, written by Daniel R. Miller, is available at the web site address: www.math.ilstu.edu/TI-92. In each figure the two triangles are images of each other under the particular involution. They are congruent when the involution is an isometry, i.e., a line reflection or the 180° rotation.

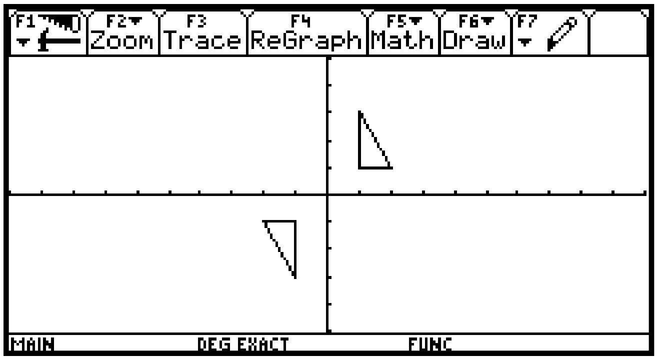

Figure 1a: $R_{180} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

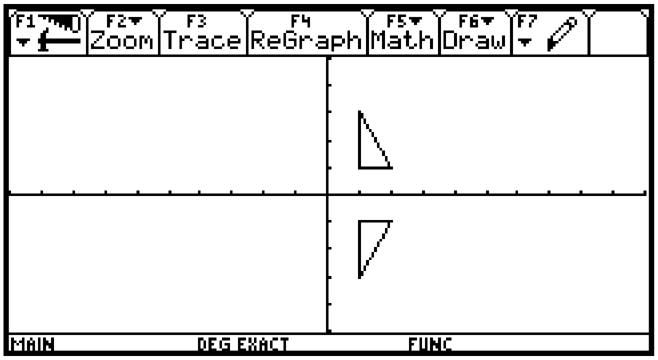

Figure 1b: $R_x = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

Case (2) b = 0, c ≠ 0. From (iii), d = -a.

From (i) and (iv), either (a, d) = (1, -1) or (a, d) = (-1, 1).

Subcase (2.1) : With (a, d) = (1, -1), A becomes $A_1 = \begin{bmatrix} 1 & 0 \\ c & -1 \end{bmatrix}$.

Letting $T_1(x) = A_1x$, where $x = \begin{bmatrix} x \\ y \end{bmatrix}$, we first find the fixed points of T_1 by solving $A_1x = x$, which is equivalent to (A - I)x = 0, or $\begin{bmatrix} 0 & 0 \\ c & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. This matrix equation shows that the line y = (c/2)x is pointwise fixed by T_1. The image of x is $x' = \begin{bmatrix} x \\ cx - y \end{bmatrix}$. The midpoint of the segment joining x and x' is (x, cx/2), indicating that the midpoint of xx' lies on the invariant line $y = (c/2)x$. Note that this is a property of ordinary reflections. Also, when x is not on the invariant line, the segment joining x and x' is vertical, independent of x. The fact that all segments xx' are parallel is another property of ordinary reflections. But, since $c \neq 0$, the segments joining points x and their images x' are not perpendicular to the invariant line, so T_1 is not an ordinary reflection. (See figure 2, where $c = 2$.) This type of mapping, which shares all of the properties of ordinary reflections except the perpendicularity property and preservation of distance, is called an oblique (or skew) reflection.

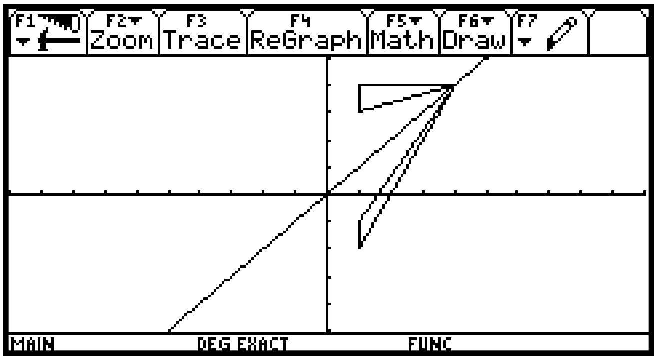

Figure 2: $A_1 = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}$

Invariant line: $y = x$

All segments xx' are vertical.

Subcase (2.2) : With $(a, d) = (-1, 1)$, A becomes $A_2 = \begin{bmatrix} -1 & 0 \\ c & 1 \end{bmatrix}$.

Letting $T_2(x) = A_2x$, we find that the line $x = 0$ is pointwise fixed and, as with T_1, all segments joining points x not on this line to their images x' are parallel (with slope $-c/2$) and are bisected by the invariant line. T_2 is also an oblique reflection. (See figure 3, where $c = 1$.)

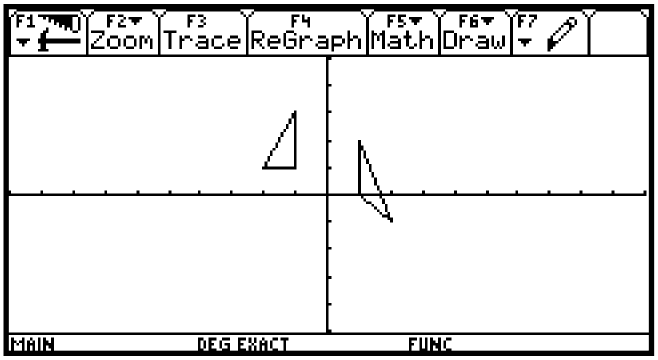

Figure 3: $A_2 = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}$

Invariant line: x = 0

All segments xx' have slope -1/2.

Case (3) $b \neq 0$, $c = 0$. From (ii), $d = -a$.

From (i) and (iv), either (a, d) = (1, -1) or (a, d) = (-1, 1).

This case is similar to case (2). The subcase matrices are $A_3 = \begin{bmatrix} 1 & b \\ 0 & -1 \end{bmatrix}$ when (a, d) = (1, -1) and $A_4 = \begin{bmatrix} -1 & b \\ 0 & 1 \end{bmatrix}$ when (a, d) = (-1, 1). The transformations T_3 and T_4 are both oblique reflections whose invariant lines are y = 0 (see figure 4, where b = -3) and y = (2/b)x (see figure 5, where b = 4), respectively. The slope of segment xx' is -2/b for T_3 and 0 for T_4.

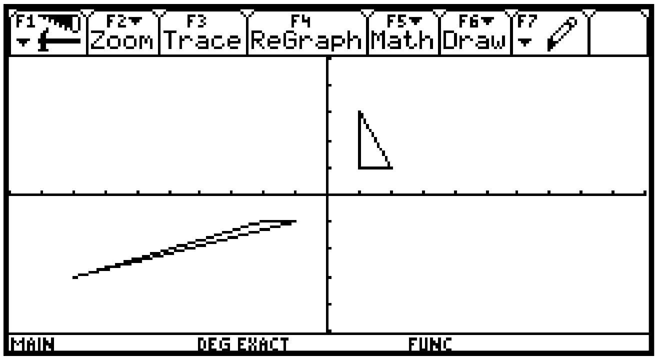

Figure 4: $A_3 = \begin{bmatrix} 1 & -3 \\ 0 & -1 \end{bmatrix}$

Invariant line: $y = 0$

All segments xx' have slope 2/3.

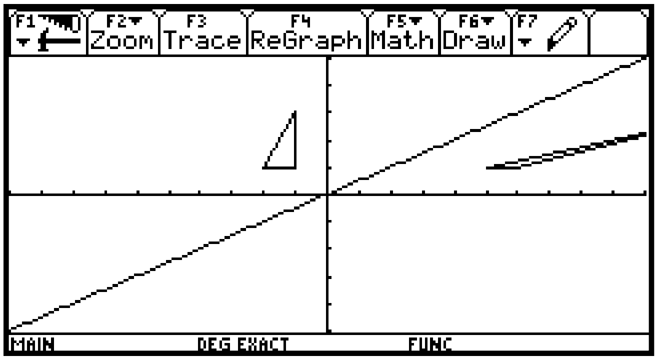

Figure 5: $A_4 = \begin{bmatrix} -1 & 4 \\ 0 & 1 \end{bmatrix}$

Invariant line: $y = (2/4)x$

All segments xx' are horizontal.

Case (4) $bc \neq 0$. Then $d = -a = \pm\sqrt{1 - bc}$, where $bc \leq 1$.

Subcase (4.1) : If $b = c = 1$, then $d = a = 0$ and the matrix A is $A_5 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, which is a special case worth noting: T_5 is reflection across the line $y = x$.

Subcase (4.2) : If $b = c = -1$, then again $d = a = 0$ and the matrix A is $A_6 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$, another special case worth noting: T_6 is reflection across the line $y = -x$.

Subcase (4.3) : $bc = 1$, but $b \neq c$. The matrix A is $A_7 = \begin{bmatrix} 0 & b \\ 1/b & 0 \end{bmatrix}$ and the line $y = (1/b)x$ is pointwise fixed. The mapping T_7 is an oblique reflection with segments xx' having the constant slope $-1/b$ when x is not on the invariant line. (See figure 6, where $b = 3$.)

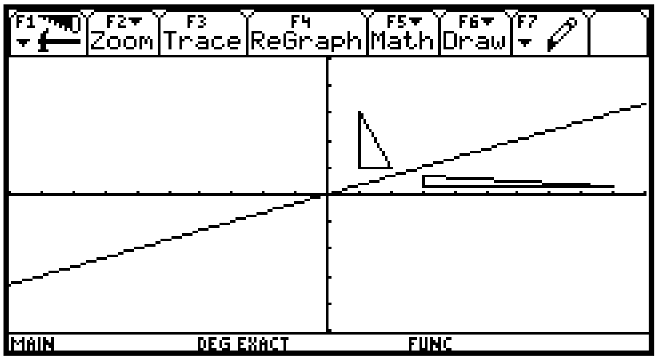

Figure 6: $A_7 = \begin{bmatrix} 0 & 3 \\ 1/3 & 0 \end{bmatrix}$

Invariant line: $y = (1/3)x$

All segments xx' have slope $-1/3$.

Subcase (4.4) : $bc < 1$. The matrix A is either $A_8 = \begin{bmatrix} \sqrt{1 - bc} & b \\ c & -\sqrt{1 - bc} \end{bmatrix}$ or $A_9 = \begin{bmatrix} -\sqrt{1 - bc} & b \\ c & \sqrt{1 - bc} \end{bmatrix}$. The transformations T_8 and T_9 are oblique reflections whose invariant lines are $y = \frac{1 - \sqrt{1 - bc}}{b}\, x$ for T_8 (see figure 7, where $b = 2$ and $c = .32$) and $y = \frac{1 + \sqrt{1 - bc}}{b}\, x$ for T_9 (see figure 8, where $b = -2$ and $c = -.375$).

For T_8 the segments xx' have the constant slope $\frac{c}{\sqrt{1 - bc} - 1}$.

For T_9 the segments xx' have the constant slope $\frac{c}{-\sqrt{1 - bc} - 1}$.

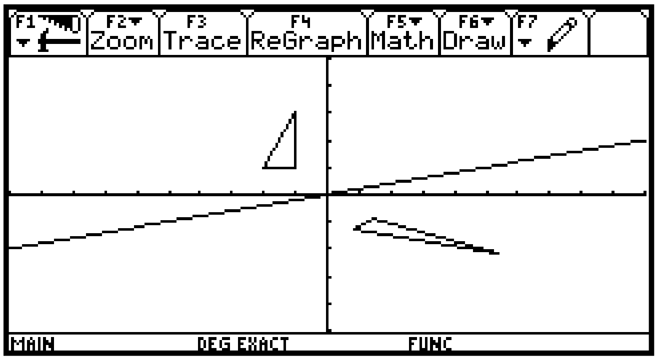

Figure 7: $A_8 = \begin{bmatrix} .6 & 2 \\ .32 & -.6 \end{bmatrix}$

Invariant line: $y = .2x$

All segments xx' have slope -.8.

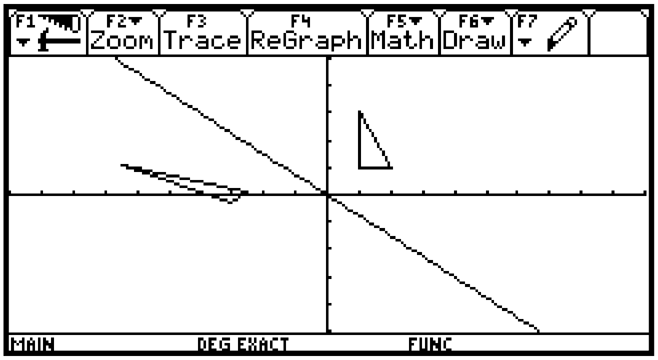

Figure 8: $A_9 = \begin{bmatrix} -.5 & -2 \\ -.375 & .5 \end{bmatrix}$

Invariant line: $y = -.75x$

All segments xx' have slope .25.

Subcase (4.5) : $b = c \neq 0, \pm 1$. The matrix A is either $A_{10} = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$ or $A_{11} = \begin{bmatrix} -\cos\theta & \sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$. The transformations T_{10} and T_{11} are ordinary reflections whose invariant lines are $y = \dfrac{(1 - \cos\theta)}{\sin\theta}\,x$ (or $y = (\tan\dfrac{\theta}{2})x$) for T_{10} and $y = \dfrac{1 + \cos\theta}{\sin\theta}\,x$ (or $y = \cot(\dfrac{\theta}{2})x$) for T_{11}.

These cases exhaust all of the possibilities, so we can conclude that a linear involution of the plane is either the identity mapping, the 180° rotation about the origin, an ordinary reflection across a line through the origin, or an oblique reflection across a line through the origin. Of these four types, only the last is likely to be unfamiliar to most students. Because of its close similarity to an ordinary reflection, the oblique reflection is an interesting one for investigation. Moreover, its appearance in this kind of analysis shows that a case-by-case breakdown can reveal some surprises. Students may then see that adding such a stategy to their repertoires may be useful in other open-ended problem situations.