A DERIVATION OF EULER'S CONSTANT

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Introduction

Many early mathematicians were intrigued by the P-Series (Euler, Leibniz, the Bernoullis, Newton,…,etc) Gottfried Wilhelm Leibniz showed that the odd alternating harmonic series summed to — . By using the inverse tangent series

(— — — ———). If x=1, then — .

This means that — — — — ——— ,see appendix A and B

Leibniz knew that the alternating harmonic series converged by his ALTERNATING SERIES TEST THEOREM.

Leibniz believed that the English mathematicians had discovered a formula for the partial sums of the harmonic series. To get this formula, Leibniz offered his derivation of — .

— — — The partial sums grow very slowly.

It almost seems that this series should converge!!! - As if it were bounded!!!!

Jakob Bernoulli proved that the harmonic series diverges.

Proof: The Harmonic Series Diverges:

For a > 0,

The terms consists of terms which decrease in size.

From the above.

Euler also had a proof that this harmonic series diverges.

Proof: The Harmonic Series Diverges:

Euler considered the natural logarithms (ln (1-x)). Euler recognized the link between logarithms and the harmonic series.

Remember the power series for:

Euler said let x = 1, then:

Note: the logarithm increases without bound.

Euler's Constant

While looking at this problem, Euler noticed something more peculiar. Euler recognized the link between logarithms and the harmonic series.

If , then the gives an interesting connection to the Harmonic Series.

If the values for n = 1, 2, 3,… , we can sum the equations and get the Harmonic Series.

Sum the above equations by adding the columns.

Another way to look at gamma.

Is γ rational or irrational???? - Nobody knows!!!

Proof that gamma exists.

Let's look at the sequence.

Notice that: which means the sequence is increasing.

The sequence is increasing and bounded above by "1".

Appendix A

is easy to derive. All we need to do is integrate the

geometric series term by term from 0 to x. Then let x = 1.

We begin with the series:

By integrating over the

interval gives

Where .

Since , .

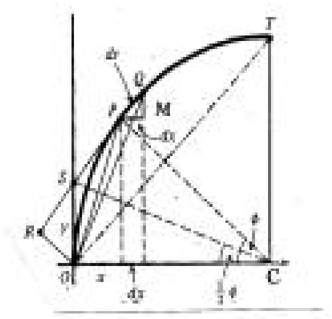
For .

In other words, , which give us the series

Therefore: If x = 1,

Appendix B

This is Leibniz's proof that the alternating odd harmonic series sums to .



Leibniz showed that the area bounded between the chord OT and the quarter of a unit circle plus the area of was:

. The area of this quarter circle is .

Leibniz first finds the area between the circular arc and the chord OT.

He does this by summing the areas of triangles with vertex "O" and opposite side "PQ", where dx ( ) intersects the circular arc. OR is drawn perpendicular to the linear line.

QPS. The area of

(note: OR is the altitude from vertex O to the extended base PQ).

As x goes from 0 to 1, many such triangles are formed and the area can be summed in an integral. Let "y" represent the length of OS, then the area of is:

. The total area (A) of the bounded sector becomes:

Leibniz used integration by parts to get:

From the diagram:

If it was difficult the see why , then substitute the above parametric equations for x and y and evaluate

From the parametric equations for x and y, the following can be derived:

. Which yields the following:

From the area of the sector equation we get:

To find the total area of the quarter (unit) circle and the area of the triangle, , needs to be added. This gives the following equation:

Leibniz thought this equation was one of his greatest achievements.

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