Hubris, Weird Numbers, a Missing Asterisk, and Paul Erdos

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by

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Die ganzen Zahlen hat Gott gemacht, andere ist Menschenwerk.

Leopold Kronecker

Outline

- Hubris
- Weird Numbers
- A Missing Asterisk
- Paul Erdos
- Open Questions

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Personal Chronology

1967
- B.A. in Mathematics from UC Riverside

1969
- M.A. in Mathematics from CSU San Diego
- Summer job at the Naval Undersea Center in San Diego
  - –Develop software to convert analog data to digital data

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Personal Chronology

1973
- Ph.D. in Mathematics (Number Theory) from The Pennsylvania State University

1973 - 1998
- Wagner Associates
  - –Search for lost objects
  - –Financial optimization
  - –Biotechnology

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Hubris

Summer of 1969

Possible Thesis Topics
- –Riemann Hypothesis
- –Fermat's Last Theorem
- –Perfect Numbers

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Review of Perfect Numbers

For n a natural number, let
$s(n) = \sum_{\substack{d|n \\ d<n}} d$.

If s(n) = n, then n is a **perfect number**.

Known to Euclid.

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The First Five Perfect Numbers

$6 = 2 \cdot 3 = 2^1(2^2 - 1)$
$28 = 2^2 \cdot 7 = 2^2(2^3 - 1)$
$496 = 2^4 \cdot 31 = 2^4(2^5 - 1)$
$8128 = 2^6 \cdot 127 = 2^6(2^7 - 1)$
$33550336 = 2^{12} \cdot 8191 = 2^{12}(2^{13} - 1)$

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Euclid and Euler

Euclid: If $n = 2^{m-1}(2^m - 1)$ and $2^m - 1$ is prime, then n is perfect. (*Elements*, Book IX, Proposition 36.)

Euler: If n is an even perfect number, then $n = 2^{m-1}(2^m - 1)$ and $2^m - 1$ is prime. (Posthumous paper.)

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Mersenne Primes

- If $p = 2^m - 1$ is prime, then p is called a **Mersenne** prime.
- If $2^m - 1$ is prime, then m is prime.
- Frank Nelson Cole (1903):

$2^{67} - 1 = (193707721)(761838257287)$

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Mersenne Primes and GIMPS

- 39 known Mersenne primes
- Last 5 found by the Global Internet Mersenne Prime Search (GIMPS)

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Mersenne Primes and GIMPS

- Largest known prime is $2^{13466917} - 1$
  - Over 4 million digits
  - "I had some yellow apples occupying a crisper."
- $50,000 prize awarded for the discovery of a million-digit prime
- $100,000 prize available for the discovery of a 10-million digit prime

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Oldest(?) Unsolved Problems

- Is there an odd perfect number?
- Are there infinitely many even perfect numbers? (Are there infinitely many Mersenne primes?)

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Some Results for Odd Perfect Numbers

If N is an odd perfect number and k is the number of distinct prime factors of N:

- $k \geq 8$
- $N < (4k)^{(4k)^{k^2}}$
- $N > 10^{300}$
- $N = p^m r$ where p is a prime not dividing r and $p \equiv m \equiv 1 \pmod 4$

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Obvious Question #1

How does a number fail to be perfect?

If $s(n) < n$, then n is **deficient**.

If $s(n) \geq n$, then n is **abundant**.

Deficient and abundant numbers first described by Nichomachus (≈ 100 A. D.)

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Obvious Question #2

If n is abundant, is there a set of proper divisors that add up to n?

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Pseudoperfect Numbers

n is **pseudoperfect** if it is the sum of distinct proper divisors.

Pseudoperfect first defined by Waclaw Sierspinski in 1965.

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Obvious(?) Question #3

- If n is abundant, is it pseudoperfect?

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Summer of 1969

- n is called a **weird** number if it is abundant, but not pseudoperfect.
- The smallest weird number is 70.
- Checked by hand up to 300.
- Wrote a FORTRAN program to find all weird numbers ≤ 100,000.

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The First 14 Weird Numbers

70 = 2 5 7	**10430** = 2 5 7 149
836 = 2^2 11 19	**10570** = 2 5 7 151
4030 = 2 5 13 31	**10792** = 2^3 19 71
5830 = 2 5 11 53	**10990** = 2 5 7 157
7192 = 2^3 29 31	**11410** = 2 5 7 163
7912 = 2^3 23 43	**11690** = 2 5 7 167
9272 = 2^3 19 61	**12110** = 2 5 7 173

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70 and 836

70 = **2 5 7**

s(70) = 1 + 2 + 5 + 7 + 10 + 14 + 35 = 74

836 = **2^2 11 19**

s(836) = 1 + 2 + 4 + 11 + 19 + 22 + 38 + 44 + 76 + 209 + 418 = 844

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4030

4030 = **2 5 13 31**

s(4030) = 1 + 2 + 5 + 10 + 13 + 26 + 31 + 62 + 65 + 130 + 155 + 310 + 403 + 806 + 2015 = 4034

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Submitted Problem to MAA Monthly (1971)

Are there any abundant numbers that are not pseudoperfect?

*Are there any odd abundant numbers that are not pseudoperfect?

The problem appeared without the asterisk.

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Correspondence with Erdos

- Received letter from Paul Erdos in October 1971
  - He and Straus missed 70
  - Erdos offered $10 for an odd weird number or $25 for a proof that none exists.
- Exchanged letters

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Erdos Visits Penn State

- Colloquium in the Fall of 1972.
- We worked together all Saturday morning.
- He asked if I wanted to publish a joint paper.

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Joint Paper

- Submitted to *Mathematics of Computation*
- Galley proofs
- Published in April of 1974

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What is Known about Weird Numbers

- The weird numbers less than 10^6 have been published.
- The weird numbers less than 10^9 are known.
- If n is weird, $2^k \mid n$, and p is an odd prime with p | n, then $2^k < p$.

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What is Known about Weird Numbers

- All known weird numbers are even.

The smallest odd abundant number is 945.

s(945) = 1 + 3 + 5 + 7 + 9 + 15 + 21 + 27 + 35 + 45 + 63 + 105 + 135 + 189 + 315 = 976

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What is Known about Weird Numbers

- The weird numbers have positive density.

Let w(n) be the number of weird numbers ≤ n, then

$\lim_{n \to \infty} \frac{w(n)}{n}$ exists and is positive.

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What is Known about Weird Numbers

- If n is weird and p is a prime with p > s(n) + n, then pn is weird.
- If p and p + 2 are prime (p > 3), then p + 1 is pseudoperfect.

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The Concept of Primitive

n is said to be **primitive** weird (abundant, pseudoperfect) if n is weird (abundant, pseudoperfect) and no proper divisor of n is weird (abundant, pseudoperfect).

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What is Known about Weird Numbers

- There are 152 primitive weird numbers less than 10^9.
- If $n = 2^{k-1}(2^k + 1)(2^{2k-1} - 1)$ and $2^k + 1$ and $2^{2k-1} - 1$ are prime, then n is primitive weird.
  - 70 and 17272 have this form.
  - The converse is not true.

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Open Questions and Cash Prizes

- Is there an odd weird number?
  - $20
- Are there infinitely many primitive weird numbers? (There are 152 less than 10^9.)
  - $25

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Open Questions

- Are there infinitely many primitive weird numbers of the form $n = 2^{k-1}(2^k + 1)(2^{2k-1} - 1)$?
  - $25
- Can $s(n)/n$ be arbitrarily large for weird n?
  - $30

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For Angling may be said to be so like the Mathematicks, that it can never be fully learnt...

Izaak Walton
The Compleat Angler