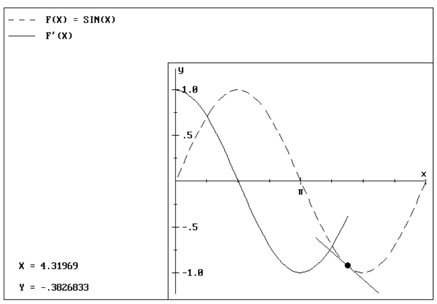
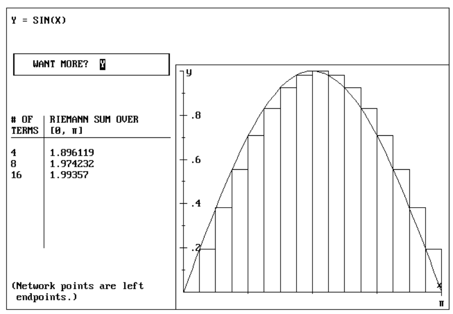
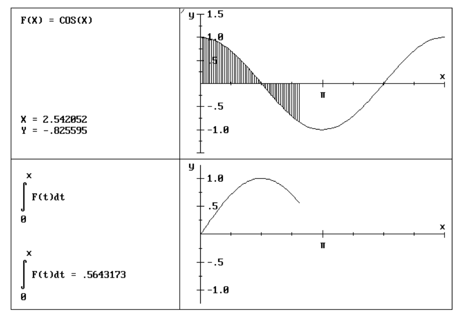
USING CONVERGE AND DERIVE

IN THE TEACHING OF CALCULUS







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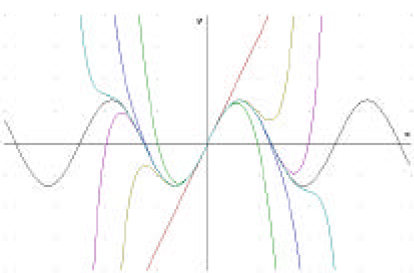

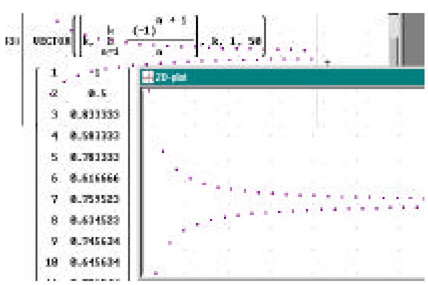

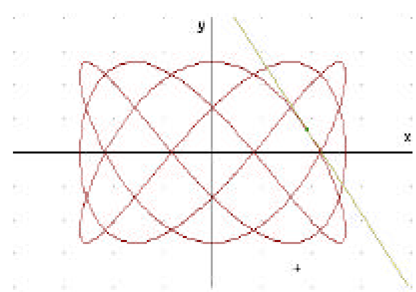

Brookdale Community College
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PREFACE

The programs we will be working with today, ***CONVERGE*** (from JEMware) and ***DERIVE*** (from the Soft Warehouse), enable us to teach Calculus in a new and very exciting way. *Converge* has a wide variety of specialized programs that let students really visualize the conceptual underpinnings of Calculus. *Derive* is the assistant that does the time-consuming mathematics for students once they have mastered the basics. Together these programs provide excellent tools for assisting students to understand the concepts of Calculus and to solve problems without regard for the level of algebraic difficulty. Our goal today is not to teach the programs, but to have you experience them as students do - in the context of Calculus. We have given a great deal of thought over the years to our curriculum and the role of technology. In 1989 we were awarded a grant from the New Jersey Department of Higher Education for our first computer classroom and have been teaching Calculus in a computer environment ever since. We now have a second computer classroom (funded partially through a grant from the National Science Foundation) so that all Calculus classes and classes from several other courses meet in this computer environment. Because we were successful in writing these grants, we have the luxury of using the computer as a tool at any time during any class. The pages that follow are a sampling of the exercises we have students do on a daily basis. You will notice that in general, the exercises incorporate writing, copying graphs and/or tables, and in some cases, calculations to be done first by the student without the aid of the computer. This is the way each class goes and it would be misleading to show you only the part of the exercises that involve the use of the computer. To be a truly successful tool, the computer programs must be smoothly integrated with lecture and with explorations on the part of the student. As you work through the exercises today, keep in mind the technology you have available at your school and you can begin to think about how best to use it.

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INTRODUCTION TO GRAPHING WITH CONVERGE

Example 1: Use *Converge* to: (a) graph $f(x) = x^4 - 6x^3 + 8x^2 + 2x - 3$, (b)produce a table of values for $-2 \le x \le 5$, and (c) estimate the zeros of the function. Use the following directions:

a. Choose **G**raph, then **G**raph with a function file. For "function file to graph with" type **deg4c** and hit **<Enter>**.
When the formula for this function is displayed, hit the **<End>** key and type **+ 2x – 3** and then hit **<Enter>**. Use this graphing window:
Min X: **–2**, Max X: **5**, Min Y: **10**, Max Y: **6**. Then hit **<F5>** to accept the rest of the settings.
Write a few sentences about why this is a reasonable graph for a polynomial of degree 4.

Looking at the graph, how many zeros does this function have?

From the graph, give the approximate location of the zeros.

b. To generate a table of values, choose **P**ost - graph, then **T**able with graph, and say **n**o to the printer. Use First X: **2** , Last X: **5**, and Increment: **.1** . Once you have the table, you can use ↑ or ↓ or **<Page Up>** or **<Page Down>** to scroll through the table.
Examine the table and find intervals that contain the zeros of the function.

c. To approximate the zeros of the function, choose **P**ost - graph, then **E**stimate function zeros and say **n**o to the printer. At "Guess for X", enter your estimate for the left-most zero. Converge will display its best approximation for this zero. Repeat this process for the remaining zeros and write your results here.

Example 2: Graph the rational function $f(x) = \dfrac{x+3}{x(x^2-4)}$ and identify all the asymptotes.

Choose **G**raph then G**r**aph without a file and choose **R**ational. Type the function
(X + 3) / (X (X^2 – 4)) and hit **<Enter>**.
Use this window: Min X: **5**, Max X: **5**, Min Y: **6**, Max Y: **6**. Examine the rest of the settings. Notice that there is an X in the box marked "Check for asymptotes". *Converge* knows that rational functions often have asymptotes and if you specify a rational function, this box automatically has an X in it. If you had chosen Y = F(X), you would have had to put an X in that box yourself to tell *Converge* to look for asymptotes. Now hit **<F5>** to accept the remaining settings.
Now generate a table of values for this function. Choose **P**ost - graph, then **T**able with graph, and say **n**o to the printer. Use First X: **5** , Last X: **5**, and Increment: **.1** .
Use the graph, the table, and the equation to answer these questions. How many vertical asymptotes does this function have? Where are they located?

What is the horizontal asymptote for this function?

Does the graph of this function ever cross its horizontal asymptote? How do you know?

What is the zero of this function?

Example 3: Graph the piecewise function $f(x) = \begin{cases} x^2+3 & \text{if } x<1 \\ -2x^2+4 & \text{if } x \geq 1 \end{cases}$.

To graph a piecewise function in *Converge*, the first function is enclosed in parentheses, followed by a FOR statement in brackets, then a semicolon, then the second function in parentheses followed by its FOR statement in brackets.

Choose **G**raph, then G**r**aph without a file, then **Y** = F(X) and **1** function. Type the function as follows: **(x ^ 2 + 3) [for x < 1] ; (–2x ^ 2 + 4) [for x >= 1]** and hit **<Enter>**.
Use this window: Min X: **3**, Max X: **3**, Min Y: **9**, Max Y: **9**. Hit **<F5>** to accept the remaining settings.

Do you notice anything wrong with this graph?

Repeatedly press **<Enter>** until you get to the option "Smallest distance for graph break checking". Notice that the default value is 9R. This has to do with the number of rows on the screen and it is easier to work with actual units. What *Converge* will do is connect plotted points unless it is instructed to inspect for vertical breaks in the graph. *Converge* needs to be given some guidance on a vertical distance that would indicate a break in the graph. A good rule of thumb is to specify a distance that is half of the actual vertical break distance. Notice that this graph has a vertical break of 2 units. Try typing a **1** in this option and then hit **<F5>**.

Was the break displayed properly this time?

Before we leave this function, try one other option. Once again, hit **<F6>** to Regraph the function and press **<Enter>** to accept the function. This time use the **<Tab>** key to get to the option "Enhance break pts". Hit the **<Spacebar>** to put an X in this box and then hit **<F5>**.

What has changed about the graph?

LIMITS

Example 1: Investigate $\lim_{x \to 0} \frac{\sin x}{x}$ graphically and numerically.

a. To investigate graphically, in *Converge*
Choose **G**raph
Choose **G**raph with a function file
Type **sinc** as the file name and hit **<Enter>**
Hit **Enter>** to accept the function and **<F5>** to accept the settings.

You can obtain a more close-up view of the graph near $x = 0$ by zooming in as follows:
Hit <**F7>** for the Zoom menu.
Choose **H**orizontal
Set the zoom factor at **.5**.

Repeat this several times until you have a good idea of the behavior of the graph as x gets very close to 0.

Describe the behavior of the graph of $f(x) = \frac{\sin x}{x}$ as x gets very close to 0.

Get the original graph back as follows:
Hit **<F7>** for the Zoom menu.
Choose **R**estore original scaling.

b. Investigate the limit numerically.
Choose **C**alculus
Choose **A**pproach a limit graphically
Choose **G**et the rule from a file and type **sinc** or hit <**Enter>** if it is already there.
Hit <**Enter>** to accept the function and **<F5>** to accept the settings.
Say **N**o to the printer, specify A = **0** and approach from the **L**eft.
Keep hitting **<Enter>** and as you do watch two things: the values of the function on the table and the points on the graph. Continue until you reach "undefined" for Y.
At one point, the Y values on the table became 1. Are these values correct? Explain.

Describe what happened numerically as x approached 0 from the left:

When you are finished, respond **N**o to Want More? and choose Approach a limit with the **s**ame function.
Keep A = **0** and approach A from the **R**ight.
Once again, hit **<Enter>** repeatedly until you reach "undefined" for Y.
Describe what happened numerically as *x* approached 0 from the right:

c. What conjecture would you make about $\lim_{x \to 0} \frac{\sin x}{x}$ based on the graphical and numerical evidence?

Example 2: Consider the function $f(x)=\begin{cases}\log_2(-x) & \text{if } x\le -2\\ 5-x^2 & \text{if } -2<x<0\end{cases}$.

a. Obtain a graph of this function. This function is a built-in function file in *Converge.*

Choose **G**raph, then choose **G**raph with a function file.
Type the name of the file: **limit12** and hit **<F9>** to accept all the settings.
and obtain the graph.

Copy this graph below.

b. Numerically investigate $\lim_{x\to -2^-} f(x)$. To do this, you can use the options **C**alculus, then **A**pproach a limit graphically, **G**et the rule from a file, and hit **<Enter>** to accept limit12. Hit **<F9>** to accept all the settings. Say **N**o to the printer, specify A = **–2**, and approach from the **L**eft. Keep hitting **<F10>** to generate values of the function as *x* gets closer and closer to –2. Copy down a representative sample of this table:

c. Now investigate $\lim_{x\to -2^+} f(x)$. To do this, hit <**Esc>** and then choose Approach a limit with the **s**ame function. Let A = **–2**, and approach from the **R**ight. Keep hitting **<F10>** to generate values of the function as *x* gets closer and closer to –2. Copy down a representative sample of this table:

d. Write down your conclusions concerning $\lim_{x \to -2^-} f(x)$, $\lim_{x \to -2^+} f(x)$, and $\lim_{x \to -2} f(x)$.

e. Determine whether $f(x)$ is continuous at $x = -2$. Justify your answer in terms of the definition of continuity

Example 3: Consider the function $f(x) = \begin{cases} 3\sin x & \text{if } x \le -1 \\ \dfrac{4x}{x^2+1} & \text{if } x > -1 \end{cases}$.

a. Obtain a graph of this function.

Choose **G**raph, then G**r**aph without a file, choose **Y = F(X)** and **1** function.
Type the function: **(3 sin(x)) [for x < = –1] ; (4x / (x^2 + 1)) [for x > –1]**
Use this window: Min X: **–5,** Max X: **4,** Min Y: **–4,** Max Y: **4** and change the Smallest distance for graph break checking to **.5**. Then hit **<F5>**.

Copy this graph here.

b. As you did in problem 1, determine whether $\lim_{x \to -1} f(x)$ exists by investigating the left and right hand limits. Provide numerical evidence as well as the graph you obtained in part **a**. To do this, you can use the options **C**alculus, then **A**pproach a limit graphically.

Summarize your conclusions here:

c. Determine whether $f(x)$ is continuous at $x = -1$. Justify your answer in terms of the definition of continuity

Example 4: Analyze the global behavior of the function $f(x) = \dfrac{-2x^3 - 3x + 5}{4x^3 + 6x^2 - 7}$ by investigating $\lim_{x \to -\infty} f(x)$ and $\lim_{x \to \infty} f(x)$.

a. Obtain a graph of this function.

Choose **G**raph, then G**r**aph without a file, choose **R**ational and **1** function.
Type the function: **(–2 x^3 – 3x + 5) / (4 x^3 + 6 x^2 –7)**
Use this window: Min X: **–8,** Max X: **8,** Min Y: **–8,** Max Y: **8** . Then hit **<F5>**.

Copy this graph here.

b. Obtain a table of values for $f(x)$ as x gets very large in the positive direction.

Choose **P**ost-graph, then **O**ne at a time menu, then **R**ide the points on the graph.

Choose **0** as the first X and **100** as the increment. Press **<Tab>** to display a table of values as x gets very large.

Summarize your observations here.

c. Obtain a table of values for $f(x)$ as x gets very large in the negative direction.

Press **<Esc>** to end the previous table and say **Y**es to another ride. Use the same X and increment but this time hold down the **<Shift>** key while you press **<Tab>.**

Summarize your observations here.

FORMAL DEFINITION OF LIMIT

Example 1: Use *Converge* to verify that $\lim_{x\to 3}(2x-1)=5$ using the definition of limit.

Choose **C**alculus, **E**psilon-delta definition of limit, **T**ype in the rule yourself.
Type **2x – 1.**
Use this window: Min X: **2**, Max X: **4**, Min Y: **3**, Max Y: **7**. Then hit **<F5>** to accept the rest of the settings.
Choose **a = 3**. L will be computed as 5, so press **<Enter>.**
First try ε **= .5**. When you do, horizontal lines will be drawn at L + ε = 5 + .5 and L – ε = 5 – .5, creating a horizontal ε**-**corridor.
To find an appropriate δ, begin with δ **= .5.** This time two vertical lines will be drawn at *a* + δ = 3 + .5 and *a* – δ = 3 – .5, creating a vertical δ -corridor.
Look at the graph. In order for δ to work, it should be true that whenever *x* is within the vertical δ-corridor, the graph should be within the horizontal ε**-**corridor.
Choose **T**est delta and Converge will tell you whether this choice of δ appears to work.
Keep trying new delta values until you find the largest possible delta that will work for this epsilon.
Once you do this, try other values of epsilon and find the largest possible value of delta that will work for that epsilon. Complete the following table:

ε	δ
.5	
.1	
.01	
.001	

You will need to use the **Z**oom option and identify the upper left and lower right corners of better viewing rectangles each time you change ε**.**

Do you see a pattern? Is there a relationship between δ and ε? What is it?

Example 2: Use the same function with a different limit point: verify that $\lim_{x \to -1}(2x-1) = -3$.

In Converge, use the same steps as before to graph this function but use this window:
Min X: −2, Max X: **0**, Min Y: **−5**, Max Y: **−1**.
Complete the same table as in the previous example.

ε	δ
.5	
.1	
.01	
.001	

Do you see a pattern? Is there a relationship between δ and ε? What is it?

Example 3: Suppose you were asked to verify that $\lim_{x \to a}(2x-1) = 2a-3$ by finding a δ that will work for any given ε. What relationship between δ and ε will work?

Example 4: Suppose you were asked to verify that $\lim_{x \to a}(mx+b) = ma+b$ by finding a δ that will work for any given ε. What relationship between δ and ε will work?

Example 5: Test your conjecture by verifying that $\lim_{x \to 0}(-5x+3) = 3$. Use Converge as in Examples 1 and 2 but use this window: Min X: **−1**, Max X: **1**, Min Y: **−2**, Max Y: **8**. This time use your conjecture in problem 4 to try to get the correct δ right away.

AVERAGE AND INSTANTANEOUS VELOCITY; SLOPES

Numerical and Graphical Approach to Average and Instantaneous Velocity

Example 1: A ball is thrown directly upward and its height, *s*, at various times *t* is measured.
A. Numerical Approach: A table of some of these values is given below:

Table 1

t	0	1	2	3	4	5
s	4	73	110	115	88	29

a. Calculate the average velocity of the ball over the intervals $0 \le t \le 1$ and $1 \le \mathrm{t} \le 2$ and $3 \le t \le 4$ and $4 \le t \le 5$.

b. Our goal is to approximate the instantaneous velocity of the ball at $t = 1$. To begin, we will use the following table to calculate the average velocity of the ball over two small intervals around $t = 1$, the intervals $.9 \le t \le 1$ and $1 \le t \le 1.1$:

Table 2

t	0.9	1	1.1
s	67.54	73	78.14

c. Shrink the time interval around $t = 1$: Use the following table to calculate the average velocity of the ball over the intervals $.99 \le t \le 1$ and $1 \le t \le 1.01$:

Table 3

t	0.99	1	1.01
s	72.4684	73	73.5284

d. Use the following table to calculate the average velocity of the ball over the intervals $.999 \le t \le 1$ and $1 \le t \le 1.001$:

Table 4

t	0.999	1	1.001
s	72.94698	73	73.05299

What is your best estimate of the instantaneous velocity at time t = 1 second?

B. Graphical Approach: Plot some of the points from the above tables to get an idea of the shape of the graph of position vs. time for the ball.

a. Use *Converge* and choose **G**raph, then P**l**ot and connect points, press **<Enter>** to accept Use **r**ectangular coordinates. Use the following window:

Min X: 0, Max X: 6, Min Y: 0, Max Y: 120 Then hit **<F5>.**

Say **N**o to the printer, and then enter all the values in Table 1 (t corresponds to x and s corresponds to y).Once you have finished, enter the two additional values in Table 2. Do not enter the values in Tables 3 or 4.

How do the average velocities you calculated on the intervals $0 \le t \le 1$ and $1 \le t \le 2$ relate to the points plotted at (0, 4), (1, 73) and (2, 110) ?

b. The equation that gives the position of this ball as a function of time is $s = -16t^2 + 85t + 4$. Graph this function on the same window. Using *Converge*, choose **P**ost-graph, then Overla**y** one graph, choose **Y = F(X)**, type in the function: **–16 x ^ 2 + 85 x + 4** and hit <**Enter**> and then <**F5**>. This graph should contain the points previously plotted but it is continuously drawn.

Sketch the graph here:

c. Now graph the function again over the interval $0.5 \le t \le 1.5$. Choose **G**raph, then G**r**aph without a file. Choose Y=F(X) and **<Enter>** to accept the function. Use this window:
Min X: **0.5**, Max X: **1.5**, Min Y: **42**, Max Y: **96** and hit <**F5**>.
Notice that the parabolic shape is no longer apparent and in fact the graph has only a slight curve to it at this close-up view.
Sketch the graph here:

d. Regraph again (hit <**F6**> to regraph and <**Enter**> to accept the function) and use this window: Min X: **0.9**, Max X: **1.1**, Min Y: **67**, Max Y: **78.5**
Notice that now the graph appears to be a straight line.
Sketch the graph here:

e. What would be the equation of this line? Use the points (0.9, 67.54) and (1, 73) to write the equation of an estimate of this line.

f. Then overlay the graph of the line onto the graph of the function. Choose **P**ost-graph, then Overla**y** one graph, choose **Y = F(X)**, and type in the equation of your line and hit <**Enter**>. Then hit <**F5**> to accept the settings.

g. Repeat this with the points (1, 73) and (1.1, 78.14). Write the equation of the line connecting these two points and overlay the graph of this line.

h. What is the slope of the graph **at** $x = 1$?

How does this relate to the instantaneous velocity of the ball at $t = 1$ second in the beginning of this example?

Example 2: Examine the function $y = 2^x$ to approximate the slope of the graph at $x = 1$.

a. Graph the function: Using *Converge*, choose **G**raph, then G**r**aph without a file, choose **Y = F(X)**, and **1** function. Type in **2 ^ x** and hit <**Enter**>. Use this window:
Min X: **–1**, Max X: **3**, Min Y: **0**, Max Y: **8** Then hit <**F5**>.

b. Now get a more close-up view: Press <**F6**> to regraph, hit <**Enter**> to accept the function, and change to this window:
Min X: **.5,** Max X: **1.5**, Min Y: **1**, Max Y: **3** Then hit <**F5**>.
Does the graph appear linear?

c. Zoom in again: Press <**F6**> to regraph, hit <**Enter**> to accept the function, and change to this window:
Min X: **.9**, Max X: **1.1**, Min Y: **1.8,** Max Y: **2.2** Then hit <**F5**>.

Does the graph appear linear?

How could you approximate the slope of the graph at $x = 1$?

Example 3: Examine the function $y = |x - 1| + 2$ to discuss the slope of the graph at $x = 1$.

a. Graph the function: Using *Converge*, choose **G**raph, then G**r**aph without a file, choose **Y = F(X)**, and **1** function. Type in **abs(x – 1) + 2** and hit <**Enter**>. Use this window:
Min X: **–2**, Max X: **3**, Min Y: **0**, Max Y: **5** Then hit <**F5**>.

b. Now get a more close-up view: Press <**F6**> to regraph, hit <**Enter**> to accept the function, and change to this window:
Min X: **.5**, Max X: **1.5**, Min Y: **1.5,** Max Y: **2.5** Then hit <**F5**>.

Does the graph appear linear?

c. Zoom in again: Press <**F6**> to regraph, hit <**Enter**> to accept the function, and change to this window:
Min X: **.9**, Max X: **1.1**, Min Y: **1.9,** Max Y: **2.1** Then hit <**F5**>.

Does the graph appear linear?

Will this graph ever appear linear at $x = 1$?

What can you say about the slope of the graph at $x = 1$?

Example 4: Investigate the derivative of $y = -16t^2 + 85t + 4$ at $t = 1$.

a. Using *Converge*, choose **C**alculus, then **D**erivatives Menu, then **L**eft-hand derivative, then **T**ype in the rule yourself. Type in the function: **–16 x ^ 2 + 85 x + 4** <**Enter**>
Use this window: Min X: **0**, Max X: **3,** Min Y: **0**, Max Y: **120**. Then hit <**F5**>.
Specify x = **1** and first value of Δx = **–1**.
Watch as the secant line is drawn and its slope is computed.
Press <**Enter**> to let Δx get smaller and the new secant line will be drawn and its slope computed.
Repeat this process until you get a FINAL ESTIMATE. Write it here:

Describe what happened as Δx got smaller.

b. Press <**Esc**> twice. This time you will look at the right hand derivative. Choose **C**alculus, then **D**erivatives Menu, then **R**ight-hand derivative, then **T**ype in the rule yourself Press <**Enter**> to accept the function and <**F5**> to accept the window. Specify x = **1** and first value of Δx = **1**.
Once again, watch as the secant lines are drawn and their slopes are computed. Keep going until you get a FINAL ESTIMATE. Write it here:

Describe what happened as Δx got smaller.

What is your best estimate of the slope of the graph at $x = 1$?

Example 5: Examine $f(x) = x^{2/3}$ and determine if it is differentiable at $x = 0$.

First, investigate the left-hand derivative:
Using *Converge*, choose **C**alculus, then **D**erivatives Menu, then **L**eft-hand derivative.
Choose **T**ype in the rule yourself, then type the function: **x ^ (2 / 3)**. Use this window:
Min X: **–1**, Max X: **1**, Min Y: **0,** Max Y: **1**. Then hit <**F5**>.
Specify derivative at x = **0**, and first Δx = **–1**.
This time use the <**F10**> key to generate five values at a time.
What are your observations about the left-hand derivative at $x = 0$?

You can press <**Esc**> twice to exit this and then choose **C**alculus, then **D**erivatives Menu, then **R**ight-hand derivative. Then go through the same steps as above. What are your observations about the right-hand derivative at $x = 0$?

Is $f(x) = x^{2/3}$ differentiable at $x = 0$?

Does the graph have a tangent line at $x = 0$?

Example 6: Investigate $f(x) = \begin{cases} 1 - x^2 & \text{if } x \le 0 \\ x^2 - 1 & \text{if } x > 0 \end{cases}$ and determine whether it is differentiable at $x = 0$.

Using *Converge*, choose **C**alculus, then **D**erivatives Menu, then **L**eft-hand derivative.
Choose **T**ype in the rule yourself, then type the function:
(1 – x ^ 2) [for x < = 0] ; (x ^ 2 – 1) [for x > 0] Use this window:
Min X: **–1**, Max X: **1**, Min Y: **–1**, Max Y: **1**. Then hit <**F5**>.
Specify derivative at x = **0**, and first Δx = **–1**.
Hit <**Enter**> for one value at a time or <**F10**> for five at a time until you get a final estimate.
What is the left-hand derivative at $x = 0$?

Press <**Esc**> twice to exit this and then go through the same steps for the right hand derivative.
What is the right-hand derivative at $x = 0$?

Is this function differentiable at $x = 0$?
Explain.

THE DERIVATIVE FUNCTION

Example 1: Suppose $f(x) = x^2 - x$. We can compute the derivative (slope of the graph of $f(x)$) at any point *a* by using the definition of the derivative at *a*:

$f'(a) = \lim_{\Delta x \to 0} \dfrac{f(a+\Delta x) - f(a)}{\Delta x}$ For this function, the equation becomes

$$f'(a) = \lim_{\Delta x \to 0} \frac{[(a+\Delta x)^2 - (a+\Delta x)] - (a^2 - a)}{\Delta x}$$

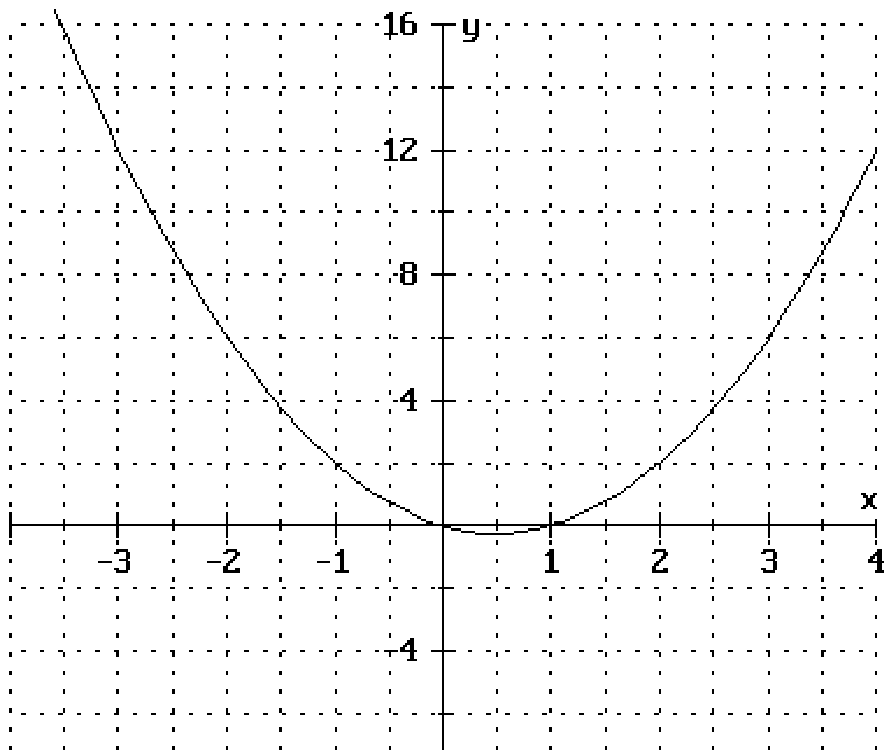
Simplify the expression for $f'(a)$.

Use your simplified expression to complete the following table.

a	–4	–3	–2	–1	0	1	2	3	4
$f'(a)$									

Choose any two ordered pairs on this table and interpret the meaning of each ordered pair as it relates to the original function:

Notice that each value of *a* generates a value of the derivative which is the slope of the graph of the function at that point. The graph of $f(x) = x^2 - x$ is given at the top of the next page. Plot the values of the derivative (the ordered pairs from your table) on this graph.



Write down any observations you have about the relationships between the graph of the points on the table and the graph of $f(x)$.

____

____

Now get into *Converge*.

Choose **C**alculus, then **D**erivatives Menu, then **1.** Graph f and f '. Say **Y**es to Tangent lines, and choose **T**ype in the rule yourself. Type in the function: **x^2 − x** and hit **<Enter>**. Use this Window:
Min X: **−4**, Max X: **4**, Min Y: −10, Max Y: **20** . Also, put an X in the box marked Freeze each graph at first point.
Hit **<F5>** and then the **<Spacebar>** to get the graph of the function.
Now hit the <**Tab>** key repeatedly and watch what happens. Converge will draw the tangent lines to $f(x)$ at values of x starting at the left endpoint. The values of x and the slope of that tangent line (the slope of the graph) will be displayed on the left side of the screen. At the same time the point $(x,\ f'(x))$ will be plotted. When you are finished you will have a graph similar to the one you produced above.

Example 2: Graph the function that is the derivative of $y = \sin x$.

In *Converge*, choose **C**alculus, **D**erivatives Menu, then **1.** Graph F and F'. Say **Y**es to tangent lines, choose **T**ype in the rule yourself. Type in the function: **sin(x)** and hit **<Enter>**. Use this Window:
Min X: **−2 <Alt>p**, Max X: **2 <Alt>p**, Min Y: **−1.5**, Max Y: **1.5** . Also, put an X in the box marked Freeze each graph at first point.
Hit **<F5>** and then the **<Spacebar>** to get the graph of the function.
Now hit the <**Tab>** key repeatedly to generate the graph of the derivative. Pay close attention to where the original graph (in red) has a positive slope, a zero slope, and a negative slope, and look at the values of the derivative graph (in blue) to see where it is positive, zero, and negative.

You may want to repeat this to better understand what is happening.

Example 3: Graph the functions that are the derivatives of $f(x) = a^x$, with a = 1.5, 2, 3, and 5.

The method is the same as Example 2. Use this window for a base of 1.5 and 2:
Min X: **−5**, Max X: **5**, Min Y: **−1**, Max Y: **30 .** After you hit the spacebar to generate the function, hit the spacebar again to get the graph of the derivative quickly.
Notice that the graphs of the derivatives of $f(x) = (1.5)^x$ and $f(x) = 2^x$ both appear to be exponential but slightly different from the original functions. Notice also that the derivative graph is **below** the graph of the function.

For a base of 3 and 5, change the ymax to **80**.
Notice that the graphs of the derivatives of $f(x) = 3^x$ and $f(x) = 5^x$ also appear to be exponential but slightly different from the original functions. Notice also that the derivative graph is **above** the graph of the function.

Now graph the derivative of $f(x) = e^x$. Notice that this graph is not only exponential but appears to **coincide** with the graph of the function.

NEWTON'S METHOD

Example 1: Approximate a solution to $x^2 = 3$ by Newton's method with an initial seed $x_0 = 5$.

Using *Converge*, choose **C**alculus, then **D**erivatives Menu, then **N**ewton's method. Choose **T**ype in the rule yourself, then type the function: **x^2 – 3** and hit **<Enter>.** Use this window: Min X: –1, Max X: 5, Min Y: –3, Max Y: 22 Then hit **<F5>**. Specify **5** as the initial guess and watch the tangent line being drawn. Press **<Enter>** to apply Newton's formula again and again and watch the graph and the table.

Describe how Newton's Method works:

-=

Although *Converge* illustrates Newton's method very nicely, the number of decimal places is limited. For better precision in calculation, we will work in *Derive*.

Example 2: Use Newton's method to approximate the solution to $3^x - x^3 = 0$ with an initial seed $x_0 = 2.5$.

Commands	**Explanation**
1. In *Derive*, **O**ptions, **P**recision, **A**pproximate, **<Enter>**	Here you set *Derive* for decimal answers.
2. Transfer, **L**oad, **D**erive, **newton <Enter>**	**NEWTON** is a file that has Newton's formula stored for you.(See description below.)
3. Author, **newton(3^x – x^3, x, 2.5, 10) <Enter>** **S**implify, **<Enter>.**	To use this file, you need to type its name, newton, and in parentheses, first the function whose zero you are seeking, then the variable in the function, then the initial seed, and finally the number of iterations.

If the list is too long to fit vertically, it will be presented horizontally and scroll off the screen. To scroll through the list, hit the → once to highlight the first number in the list and then → repeatedly to scroll through the list. You can also hit the **<End>** key to see the last number in the list and the **<Home>** key to get back to the beginning of the list.

Explanation of the "newton" file

The **newton** file makes use of *Derive's* **Iterates** command, which takes the form

ITERATES(u, x, x_0, n)

where u is the expression you want to iterate, x identifies the variable, x_0 is the seed (initial value), and n is the number of iterations desired. To iterate Newton's formula, we write a new function

NEWTON(u, x, x0, n) := ITERATES(x – u/dif(u, x),x, x0,n)

Here the **iterates** command has been used to iterate the expression $x - \frac{u}{du/dx}$. To use **newton**, you need to know the function whose zero you seek, the variable is x, the initial value (seed) is x_0 and n is the number of iterations.

This function has been saved as a Derive file, called **newton.mth**. To obtain it, type **T** (for Transfer), **L** (for Load), **D** (for Derive), type **newton** and hit **<Enter>**. Once this appears on screen, you can **A**uthor newton(function, variable, seed, number of iterations) and then simplify the expression. **Be sure you are in Approximate Precision!** If the list is too long to fit vertically, it will be presented horizontally and scroll off the screen. To scroll through the list, hit the → once to highlight the first number in the list and then → repeatedly to scroll through the list. You can also hit the **<End>** key to see the last number in the list and the **<Home>** key to get back to the beginning of the list.

ANALYSIS OF A GRAPH; A DERIVE EXERCISE

Example: Analyze the graph of $f(x) = \dfrac{x+3}{x(x^2-4)}$. Include in your analysis: domain, asymptotes, critical values, local extrema, inflection points and concavity. Give a careful and accurate sketch.

We will use *Derive* to assist us in this analysis since it can differentiate functions and solve equations.

Commands	**Explanation**
1. **A**uthor **f(x):= (x+3)/(x(x^2−4)) <Enter>**	This defines the expression as the function $f(x)$.
2. **P**lot, **B**eside, **<Enter>**	This opens a graphing window in window 2.
3. **S**cale, **5 <Tab> 1 <Enter>**, **P**lot	In window 2, you have changed the viewing window. Initially, each tic mark represented 1 unit; now each tic mark on the *x*-axis represents 5 units. You have drawn the graph.

What is the domain of $f(x)$?

What are the vertical asymptotes of $f(x)$?

4. <F1> Calculus, **D**ifferentiate, **<Enter>, <Enter>, <Ctrl><Enter>**	This will give you the first derivative of $f(x)$. Hitting <Ctrl> <Enter> avoids the necessity of having to simplify the derivative in a separate step.

Make note of the fact that the derivative is expression #2.
Write $f'(x)$ here:

5. Options, **P**recision, **M**ixed, **<Enter>**	Whenever you first start *Derive*, it is in *Exact* precision, which means that it looks only for exact answers. You have changed to *Mixed* precision, where it will look for exact answers but will also try to find approximate (irrational) answers as well. You have also set the number of significant digits to 6. *Derive* also has an Approximate precision mode, where it searches only for approximate solutions.
6. so**L**ve, **<Enter>**	You have asked *Derive* to solve the equation $f'(x) = 0$; i. e. find the critical values. Notice that it returns three meaningful numbers

Write down all three critical values here:

We now need to test the critical values. We could use the first derivative test but since we will look for inflection points and need the second derivative anyway, we will use the second derivative test. First, we need to find the second derivative.

7. Calculus, **D**ifferentiate, **2**, **<Enter>, <Enter>, <Ctrl><Enter>**

Here you have found the first derivative of $f'(x)$, which is the second derivative of $f(x)$.

Make note of the fact that the second derivative is expression #9.

Write down $f''(x)$ here:

To use the Second Derivative Test on the critical values, we need to determine the sign of the second derivative at each critical value.

8. Manage, **S**ubstitute, **9 <Enter>, –4.15194 <Ctrl><Enter>**

This substitutes the first critical value into $f''(x)$. Hitting <Ctrl><Enter> avoids having to simplify the expression you would get by just hitting <Enter>.

9. Manage, **S**ubstitute, **7 <Enter>, –1.38868 <Ctrl><Enter>**
Manage, **S**ubstitute, **7 <Enter>, 1.04062 <Ctrl><Enter>**

The last step is repeated for the remaining critical values.

Summarize your conclusions regarding each critical value:

The *x*-values only *locate* the local extrema. To find the values of the extrema, we need to evaluate the original function, $f(x)$, at each critical value. Recall that the function is expression #1.

10. Manage, **S**ubstitute, **1 <Enter>, –4.15194 <Ctrl><Enter>**
Manage, **S**ubstitute, **1 <Enter>, –1.38868 <Ctrl><Enter>**
Manage, **S**ubstitute, **1 <Enter>, 1.04062 <Ctrl><Enter>**

Write down all the local extrema and identify the type.

Next we will look for inflection points. First we will set the second derivative equal to zero and solve the resulting equation. Recall that the second derivative is expression #9.

11. so**L**ve, **9 <Enter>**

Notice that *Derive* returns only one meaningful solution.

Write down all the zeros of the second derivative here:

To be sure that these are, in fact, inflection points, we should check the sign of $f''(x)$ to the left and right of this value and to the left and right of each vertical asymptote. Make an interval table indicating each of these intervals. There should be five intervals to test. Notice that when you used the second derivative test, you found the sign of the second derivative on three of these five intervals. Fill in these signs. Evaluate the second derivative at any number of your choice in the remaining interval and fill in that sign on the table.

To identify inflection points, you will need to find the *y*-values by substituting into $f(x)$. Do this as outlined in step 10 above.

Write down your conclusions concerning inflection points and concavity.

There is one final thing to investigate. Notice that this is a rational function in which the degree of the numerator is less than the degree of the denominator. Therefore, there is a horizontal asymptote.

What is the horizontal asymptote?

Notice that this graph crosses the horizontal asymptote on the left and approaches the asymptote from above as $x \to -\infty$.

Draw a careful sketch of the function and its asymptotes here:

GRAPHING PARAMETRIC EQUATIONS

Example 1: Graph $x = t,\ y = t^2$

Directions for using *Converge*:

Graph, G**r**aph without a file, P**a**rametric, **1** function.
Enter the function: X = **t**, Y = **t^2** and hit the **<F5>** function key.
Use this Window: Min X: **−8** Max X: **8** Min Y: **−8** Max Y: **70** Min T: **−8** Max T: **8**
Put an X in the box marked Freeze graph at first point. Then hit **<F5>**.
To obtain the graph, hit the **<Tab>** key repeatedly.

Copy the graph here and use arrows to indicate the direction of the path traced out.

Example 2: Graph $x = -t,\ y = t^2$

Hit the **<F6>** function key to Regraph.
Enter the function: X = **−t**, Y = **t^2** **<F5>**
Hit **<F5>** to accept the window and settings.
Use the **<Tab>** key as before to see the graph.

Copy the graph here and use arrows to indicate the direction of the path traced out.

Explain how this graph differs from problem 1.

Example 3: Graph $x = t^2, \; y = t^4$

Hit the **<F6>** function key to Regraph.
Enter the function: X = **t^2,** Y = **t^4 <F5>**
Hit **<F5>** to accept the window and settings.
Use the **<Tab>** key as before to see the graph (watch the X and Y coordinates at the start.)

Copy the graph here and use arrows to indicate the direction of the path traced out.

Explain how this graph differs from problems 1 and 2.

Example 4: Graph $x = \sin t, \; y = \sin^2 t$

Hit **<F6>**. Enter the function: X = **sin(t),** Y = **(sin(t))^2 <F5>**
In the Window, change the T-settings: Min T: **0** Max T: **2 <Alt> p.** Then hit **<F5>**.
Use the **<Tab>** key to view the graph.

Copy the graph here and use arrows to indicate the direction of the path traced out.

Explain why this graph is so limited.

Example 5: Graph $x = \sin^2 t,\ \ y = \sin^4 t$

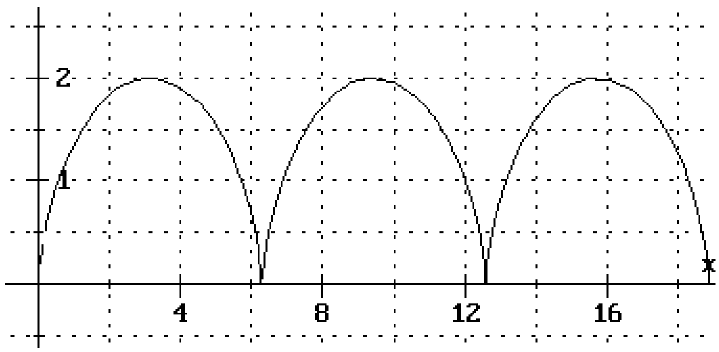
Hit **<F6>**. Enter the function: **X = (sin(t))^2** , Y = **(sin(t))^4 <F5>**
Hit **<F5>** to accept the window.
Use the **<Tab>** key to view the graph.

Copy the graph here and use arrows to indicate the direction of the path traced out.

Explain how this graph differs from problem 4.

DERIVATIVES AND PARAMETRIC EQUATIONS

Example 1: The curve traced out by a point P on the circumference of a circle as the circle rolls along a straight line is called a **cycloid** (see the figure below).



If the circle has radius r and rolls along the x-axis and the parameter is the angle of rotation θ, then the parametric equations for the cycloid are:

$$x = r(\theta - \sin\theta),\ \ y = r(1 - \cos\theta)$$

a. If $r = 1$, these equations are $x = \theta - \sin\theta$, $y = 1 - \cos\theta$. Use *Derive* to graph the cycloid for $0 \le \theta \le 2\pi$.

In *Derive*,

Author, **[t – sint, 1 – cost] <Enter>**	In *Derive*, parametric equations are entered as a *vector* , or list, of the form $[x(t), y(t)]$.
Plot, **B**eside, **<Enter>**	A graphing window is now Window 2.
Range, **–1** (Use the <Spacebar> to clear out extra digits) **<Tab> 12** (clear digits) **<Tab> –2 <Tab> 4 <Enter>**	The graphing window now shows $-1 \le x \le 12$ and $-2 \le y \le 4$
Plot, **0, 4 <Alt> p <Enter>**	t will vary from 0 to 4π.

b. Write the equation of the tangent line to the cycloid where $t = \frac{5\pi}{4}$.

Although this is easy to do by hand, we will use *Derive* for practice.

Hit **<F1> .**	The <F1> key acts as a toggle between the algebra window and the graphing window.
Manage, **S**ubstitute, **<Enter>, 5<Alt>p / 4 <Enter>**	Manage, Substitute is the way you can substitute a value into an expression in *Derive*.
Simplify, **<Enter>**	We now have the exact value of x and y.
appro**X**imate, **<Enter>**	This gives the decimal approximations.
<F1>, Plot	The point where $t = \frac{5\pi}{4}$ is plotted.

Now that we have the point (x, y), we need the slope at that point. Since the slope at a point is the derivative at that point, we need to find $\frac{dy}{dx}$ when $t = \frac{5\pi}{4}$. Since $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$, we should first find $\frac{dx}{dt}$ and $\frac{dy}{dt}$.

<F1> , **<Home>**	<Home> highlights expression #1.
Calculus, **D**ifferentiate, **<Enter>**, <**Enter>** **<Enter>**	*Derive* will take the first derivative of the highlighted expression.
Simplify, **<Enter>**	We now have the vector $\left[\frac{dx}{dt}, \frac{dy}{dt}\right]$.
$\rightarrow \rightarrow$	$\frac{dy}{dt}$ is highlighted.
Author, **<F4> /** $\leftarrow$ **<F4> <Enter>**	The <F4> key brings down the highlighted expression in parentheses. The left arrow highlights $\frac{dx}{dt}$. We have authored $\frac{dy}{dt}\Big/\frac{dx}{dt}$.
Simplify, **<Enter>**	We now have the expression for $\frac{dy}{dx}$.
Manage, **S**ubstitute, **<Enter>, 5<Alt>p / 4** **<Enter>**	$\frac{5\pi}{4}$ is substituted into $\frac{dy}{dx}$.
Simplify, **<Enter>**	This is the exact value of $\frac{dy}{dx}$ at $x = \frac{5\pi}{4}$.
appro**X**imate, **<Enter>**	This is the approximate decimal value.

Now we know that the slope is approximately $-.41423$ and the point (x, y) is approximately (4.63409, 1.70710), so using the point-slope formula, the equation of the tangent line is

$y - 1.70710 = -.414213(x - 4.63409)$ or $y = -.414213(x - 4.63409) + 1.70710$.

Author, **−.414213(x − 4.63409) + 1.70710** **<Enter>**	
<F1>, Plot, **<Enter>**	The line should be tangent to the cycloid and pass through the point previously plotted.

Sketch the graph with the tangent line here:

Example 2: Parametric equations of the form $x = a\sin\omega_1 t,\ y = b\cos\omega_2 t$ where $a, b, \omega_1,$ and ω_2 are constants, occur in electrical theory and their graphs are called **Lissajous figures.** Graph the Lissajous figure given by $x = \sin(3t),\ y = \cos(5t)$ and write and graph the equation of the tangent to the graph when $t = \frac{\pi}{12}$.

In Window 2, **W**indow, **D**esignate, **2**D-plot	The graphing window is returned to its default settings
<F1>, Remove, **1 <Enter>**	All expressions are removed.
Author, **[sin(3t), cos(5t)] <Enter>**	
<F1>, Plot, **0 <Tab> 2 <Alt> p <Enter>**	t is allowed to vary from 0 to 2π.

Clearly the curve is a very beautiful, yet complicated one. We need to locate the point where $t = \frac{\pi}{12}$.

<F1>, Manage, **S**ubstitute, **<Enter>,** **<Alt> p / 12 <Enter>**	
Simplify, **<Enter>**	This gives the exact coordinates.
appro**X**imate, <**Enter>**	This gives the approximate coordinates.
<F1>, Plot	The point is plotted.

Now find $\left[\frac{dx}{dt}, \frac{dy}{dt}\right]$ and $\frac{dy}{dx}$ as you did in the last example. Then substitute $\frac{\pi}{12}$ for t in $\frac{dy}{dx}$ to find the slope of the tangent line. Once you have the slope, write the equation of the tangent line, **A**uthor it, and **P**lot the graph.

AREA OF A PLANE REGION - RIEMANN SUMS

Example 1: Approximate the area of the region bounded by $y = x^2$, the x-axis, $x = 1$, and $x = 2$.

In *Converge*, choose **C**alculus, then **I**ntegrals Menu, then Approach an integral with **R**iemann sums. Choose **T**ype in the rule yourself, type **x^2**, and hit **<Enter>.** Use this window:

Min X: **0**, Max X: **3**, Min Y: **0**, Max Y: **6** and then hit **<F5>**.

Choose Left endpoint: **1**, Right endpoint: **2,** and finally choose endpt with **S**maller y-val.

The graph show four inscribed rectangles and the table gives the total area of the rectangles as an approximation to the area of the region.

Hit **<Enter>** and the number of rectangles will double and the table will give the total area of these rectangles as a (better) approximation to the area of the region.

Continue to press **<Enter>** and the number of rectangles will keep doubling until you reach 1024 rectangles. Copy the table here:

Write down your impressions of the way the rectangles approximated the region as the number of rectangles increased.

Now press any key and then choose Approach integral with **s**ame function and different interval. Choose the same endpoints but now choose endpt with lar**g**er y-val. This time, circumscribed rectangles are drawn and the total area of the rectangles is displayed in the table. Complete the table and copy it here:

Examine the two tables and give your best estimate of the area of the region.

ACCUMULATION FUNCTIONS : THE CONNECTION BETWEEN THE ANTIDERIVATIVE AND THE DEFINITE INTEGRAL

Example 1: Set up and graph the accumulation function for $F(x) = 2x$ over the interval [–2, 2].

In *Converge*, choose **C**alculus, then **I**ntegrals menu, then **G**raph function and antiderivative. Choose **T**ype in the rule yourself, type **2x** and hit **<Enter>**.
Use this window: Min X: −2, Max X: 2, Min Y: −4, Max Y: 4 and hit <**F5>**.
A split screen appears with the function $F(x) = 2x$ drawn in the upper window. The lower window is for the graph of the area function $A(x) = \int_{-2}^{x} 2\,t\,dt$. Since the graph is frozen, press the **<Tab>** key to get one value at a time. As you do, the *x*-value increases and the net accumulation up to that point is computed. The value of this accumulation is plotted vs. *x* in the bottom window.

Compile a table of values for *x* and A(*x*) .

Study the relationship between *x* and the accumulation function. Try to guess what function the accumulation function is. Also, look at the graph of the accumulation function and think about what its derivative would look like. Write down your observations here.

Example 2: Set up and graph the area function for $F(x) = x^2$ over the interval [–3, 3].
In *Converge*, choose **C**alculus, then **I**ntegrals menu, then **G**raph function and antiderivative. Choose **T**ype in the rule yourself, type **x^2** and hit **<Enter>**.
Use this window: Min X: −3, Max X: 3, Min Y: 0, Max Y: 9 and hit <**F5>**.

As you did for problem 1, compile a table of values for *x* and A(*x*) .

Also as you did in problem 1, study the relationship between *x* and the accumulation function. Try to guess what function the accumulation function is. Also, look at the graph of the accumulation function and think about what its derivative would look like. Write down your observations here.

Example 3: Set up and graph the area function for $F(x) = \cos x$ over the interval [0, 2π].

In *Converge*, choose **C**alculus, then **I**ntegrals menu, then **G**raph function and antiderivative. Choose **T**ype in the rule yourself, type **cos(x)** and hit **<Enter>**.
Use this window: Min X: 0, Max X: 6.5, Min Y: –1.5, Max Y: 1.5 and hit <**F5>**.

As you did for problem 1, compile a table of values for *x* and A(*x*) .

Also as you did in problem 1, study the relationship between *x* and the accumulation function. Try to guess what function the accumulation function is. Also, look at the graph of the accumulation function and think about what its derivative would look like. Write down your observations here.

Example 4: Set up and graph the area function for $F(x) = \frac{1}{x}$ over the interval [1, 3].

In *Converge*, choose **C**alculus, then **I**ntegrals menu, then **G**raph function and antiderivative. Choose **T**ype in the rule yourself, type **1/x** and hit **<Enter>**.
Use this window: Min X: 1, Max X: 4, Min Y: 0, Max Y: 2 and hit <**F5>**.

As you did for problem 1, compile a table of values for *x* and A(*x*) .

Also as you did in problem 1, study the relationship between *x* and the accumulation function. Try to guess what function the accumulation function is. Also, look at the graph of the accumulation function and think about what its derivative would look like. Write down your observations here.

AREA OF A PLANE REGION

Area of a region bounded by two curves.

Example: Find the area of the region bounded by $f(x) = e^{-x^2}$ and $g(x) = x^3 - x + 1$.

Before beginning this example, **R**emove all expressions from the Algebra window and set the graphing window to its default settings by **W**indow, **D**esignate, **2**D-plot.

Author **<Alt>e^(–x^2) <Enter>**

<F1> Plot

<F1> Author **x^3–x+1 <Enter>**

<F1> Plot

Copy the graph here and shade in the region.

We see that there are two closed regions bounded by the two curves and the *x*-values of the endpoints of these regions are determined by the points of intersection. Notice that (0, 0) is one of the points of intersection. To find the other two points, we need to solve the equation $e^{-x^2} = x^3 - x + 1$ or $e^{-x^2} - x^3 + x - 1 = 0$.

Author **<Alt>e^(–x^2)–x^3+x–1 <Enter>**

so**L**ve **<Enter>** — Notice that *Derive* is not able to find any solutions in exact precision. To find the other two values, we need to change to approximate precision.

Options, **P**recision, **A**pproximate, **<Enter>**

Notice that the two remaining solutions are in the intervals (–2, –1) and (.1, 1).

so**L**ve, **3 <Enter> –2 <Tab> –1 <Enter>** — The left-hand point is –1.27663.

so**L**ve, **3 <Enter> .1 <Tab> 1 <Enter>** — The right-hand point is 0.676227.

We now have the limits of integration. The area of a region between two curves is given by $\int_a^b (top - bottom)\,dx$. We will treat the two regions separately. In the left-hand region, the top curve is $y = x^3 - x + 1$, the bottom curve is $y = e^{-x^2}$, and the interval is [–1.27663, 0], so the integral that gives the area of this region is $\int_{-1.27663}^{0} \left(x^3 - x + 1 - e^{-x^2}\right) dx$.

Calculus, **I**ntegrate, **2–#1 <Enter> <Enter>** **–1.27663 <Tab> 0 <Enter>**	$x^3 - x + 1$ is expression #2 and e^{-x^2} is expression #1.
Simplify **<Enter>**	The value of the integral is 0.604175.

In the right-hand region, the top curve is $y = e^{-x^2}$ and the bottom curve is $y = x^3 - x + 1$ and the interval is [0, 0.676227], so the integral that gives the area of this region is $\int_{0}^{0.676227} \left(e^{-x^2} - x^3 + x - 1\right) dx$.

Calculus, **I**ntegrate, **1–#2 <Enter><Enter>** **0 <Tab> 0.676227 <Enter>**	
Simplify **<Enter>**	The value of this integral is 0.0860169.

Finally, the total area of the region is the sum of the two individual areas.

Author, **#8+#10 <Enter>**	This assumes the answers to the two integrals are on lines #8 and #10.
Simplify **<Enter>**	The sum is 0.690192.

Symbolically, we have

$$\text{A} = \int_{-1.27663}^{0.676227} \left| x^3 - x + 1 - e^{-x^2} \right| dx = \int_{-1.27663}^{0} \left(x^3 - x + 1 - e^{-x^2}\right) dx + \int_{0}^{0.676227} \left(e^{-x^2} - x^3 + x - 1\right) dx$$

$$\approx 0.604175 + 0.0860169$$

$$= 0.690192 \text{ square units.}$$

VOLUMES OF SOLIDS OF REVOLUTION
METHOD OF DISKS

We will use *Converge* to see how the volume of a solid of revolution can be approximated by the method of disks.

Example 1: Approximate the volume of the solid of revolution generated by revolving the region bounded by $y = x^2$, the x-axis, x = 1, and x = 2 about the x-axis.

In *Converge*, choose **C**alculus, then **I**ntegrals Menu, then approximate volume of solid generated by revolving the region between a graph and a **h**orizontal line.

Choose **T**ype in the rule yourself, and type **x^2** and hit **<Enter>.** Use this window:

Min X: **1,** Max X: **2**, Min Y: **−4**, Max Y: **4** and hit **<F5>**.

Axis of revolution: Y = **0** then hit **<Enter>** twice.

Left endpoint: **1**, Right endpoint: **2**, and **L**eft endpoint for network point.

Four disks will be generated and the total volume of the disks approximates the volume of the solid of revolution.

Copy down the table and keep hitting **<Enter>** to increase the number of disks and obtain a better approximation to the volume of the solid.

In each case, each disk has radius x_i^2 and height Δx, so the volume of an individual disk is $\pi\left(x_i^2\right)^2 \Delta x$. Since there are n disks and we are adding the volumes together, we have

$\text{V} \approx \sum_{i=1}^{n} \pi\left(x_i^2\right)^2 \Delta x$. Better and better approximations are obtained by letting n get larger and larger

so that we have $\text{V} = \lim_{n\to\infty} \sum_{i=1}^{n} \pi\left(x_i^2\right)^2 \Delta x = \pi \int_1^2 \left(x^2\right)^2 dx$. Evaluating this integral gives

$$\text{V} = \pi \frac{x^5}{5}\Big|_1^2 = \pi\left(\frac{32}{5} - \frac{1}{5}\right) = \frac{31}{5}\pi \approx 19.47787445 \text{ cubic units.}$$

WASHERS

Example 2: Consider the region bounded by $y = 5-(x-2)^2$ and $y = 1$. Form a solid of revolution by revolving this region about the x-axis and find the volume of the solid.

Draw the graph of the **region**:

Now use *Converge* to look at this solid of revolution and approximate its volume:

In *Converge*, choose **C**alculus, then **I**ntegrals Menu, then approximate volume of solid generated by revolving the region **b**et. two graphs about a horizontal line.
Choose **T**ype in the rule yourself, and type **5 − (x − 2)^2,** hit **<Enter>,** type **1** and hit **<Enter>.** Use this window:
Min X: **−1,** Max X: **5**, Min Y: **−6**, Max Y: **6** and hit **<F5>**.
Axis of revolution: Y = **0** then hit **<Enter>** twice.
Left endpoint: **0**, Right endpoint: **4**, and **L**eft endpoint for network point.

Four disks will be generated and the total volume of the disks approximates the volume of the solid of revolution. This time, however, notice that each disk has a hole in the center. Disk of this type are called **washers**. To see the washers better, press the **<Spacebar>** immediately after hitting **<Enter>** to ask for more disks. This will freeze the disks and you can then use the **<Tab>** key to generate washers one at a time.

To actually compute the volume of this solid, we draw representative strips in the region. These strips are perpendicular to the x-axis and when revolved around the x-axis, they form *washers*. The volume of the solid will be

$$V = \pi \int_0^4 \left((\text{outer radius})^2 - (\text{inner radius})^2 \right) dx$$

The outer radius is the distance from the x-axis to the outer curve, $y = 5-(x-2)^2$, so the outer radius is $y = 5-(x-2)^2$.
The inner radius is the distance from the x-axis to the inner curve, $y = 1$, so the inner radius is $y = 1$**.**

We now have $V = \pi \int_0^4 \left(\left(5-(x-2)^2\right)^2 - 1^2 \right) dx = \pi \int_0^4 \left(24 - 10(x-2)^2 + (x-2)^4\right) dx$

$$= \pi \left(24x - \frac{10(x-2)^3}{3} + \frac{(x-2)^5}{5} \right)\Bigg|_0^4$$

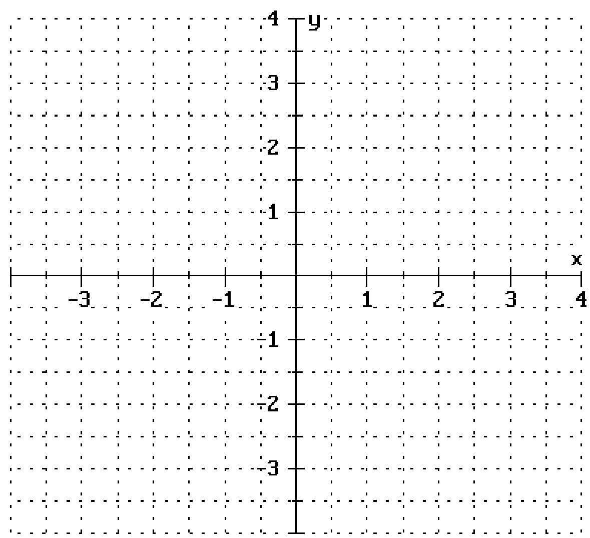
$$= \pi \left(\left(96 - \frac{80}{3} + \frac{32}{5} \right) - \left(\frac{80}{3} - \frac{32}{5} \right) \right) = \frac{832}{15}\pi \approx 174.25 \text{ cubic units}$$

DIFFERENTIAL EQUATIONS

Example 1: Given the differential equation $\frac{dy}{dx} = x^2 + 1$ and the initial condition $y(1.5) = 2.5$, solve the differential equation both graphically and algebraically.

Graphically, this equation says that the slope of the solution function at each *x* value is given by $x^2 + 1$. We can first make a table of values in which we choose some *x*-values and calculate the slope at that *x*-value. We can then construct a graph and at each *x*-value, plot a small line segment with that slope. Complete the following table and use it to construct the *slope field* on the coordinate system.

x	*dy*/*dx*
–3	
–2.5	
–2	
–1.5	
–1	
–.5	
0	
.5	
1	
1.5	
2	
2.5	
3	



Once you have the slope field drawn, locate the point representing the initial condition, (1.5, 2.5). You should be able to draw an approximate graph of the solution to the differential equation.

We can also use *Converge* to obtain both the slope field and the solution with the given initial condition. Here are the directions:

In *Converge*, choose **C**alculus, then **G**raph solutions of 1st order diff. eq..
Enter the F'(X): F'(X) = **x^2 + 1 <Enter>**
Use this window: Min X: **–4** Max X: **4**, Min Y: **–4** Max Y: **4** Then hit **<F5>**.
Compare the slope field to the one you drew.
Then type in the initial condition: x = **1.5**, y = **2.5** and compare the solution to your graph.

Algebraically, we can use a technique called *separation of variables*. In this method, *dy* and *dx* are treated as separate quantities and the differential equation is rewritten so that all *y* terms are grouped with *dy* and all *x* terms are grouped with *dx*. The object is to transform an equation of the form $\frac{dy}{dx} = G(x, y)$ into an equation of the form $f(y)dy = g(x)dx$. For this differential equation, we start with

$$\frac{dy}{dx} = x^2 + 1$$

Multiplying both sides by *dx*, we get $dy = (x^2 + 1)dx$

Now we can integrate both sides: $\int dy = \int (x^2 + 1)dx$

which gives $y + C_1 = \frac{x^3}{3} + x + C_2$

If we let $C = C_2 - C_1$, then we have $y = \frac{x^3}{3} + x + C$

The initial condition is *x* = 1.5, *y* = 2.5. Substituting, we get $2.5 = \frac{1.5^3}{3} + 1.5 + C$ or

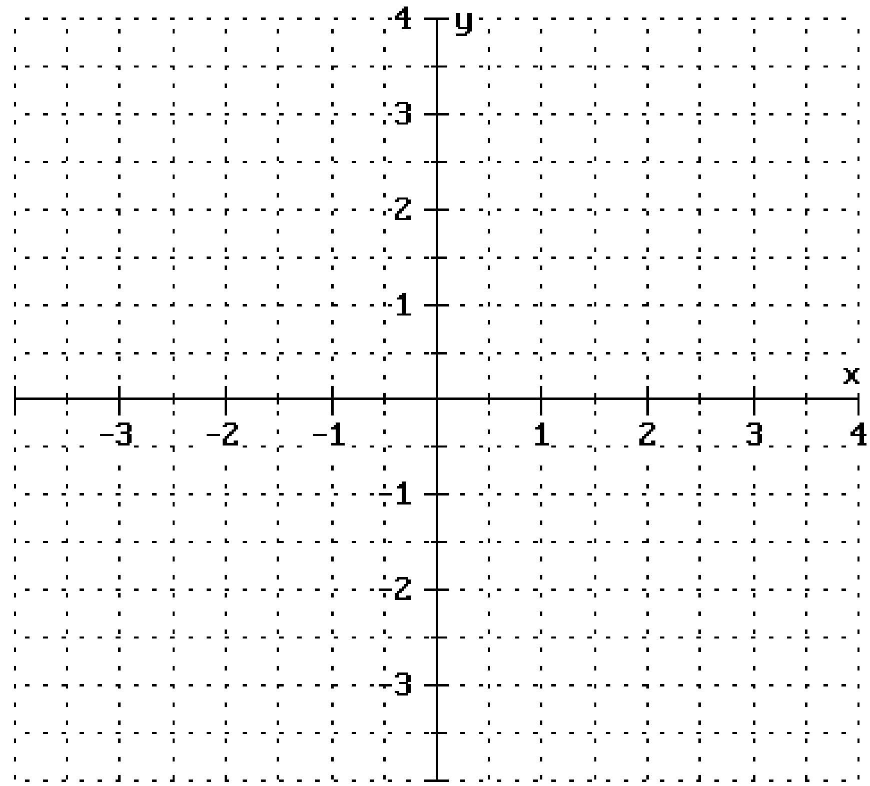
C = −.125. Thus, $y = \frac{x^3}{3} + x - .125$ is the solution that satisfies the differential equation with the specified initial condition.

We can use *Converge* to verify this solution. You should still have the slope field and the solution graph on your screen. Press **<Esc>** and then choose **P**ost-graph, then Overla**y** one graph. Choose **Y = F(X)** and type **x^3/3 + x − .125 <Enter>.** Hit **<F5>** to accept the choices. The graph should overlay exactly the previous graph.

Example 2: Given the differential equation $\frac{dy}{dx} = -\frac{x}{y}$ and the two initial conditions $y(0) = 3$ and $y(0) = -3$, solve the differential equation both graphically and algebraically.

Graphically, we will use the same strategy as we did in Example 1. This time, however, notice that the value of $\frac{dy}{dx}$ depends on both *x* and *y*, so we can only construct a table of slopes for ordered pairs (*x*, *y*). Complete the following table of slopes for the ordered pairs listed and then construct the slope field on the coordinate system.

(x, y)	dy/dx
$(0, y)$	
(1, 1)	
(1, 2)	
(1, 3)	
(1, –1)	
(1, –2)	
(1, –3)	
(2, 1)	
(2, 2)	
(2, 3)	
(2, –1)	
(2, –2)	
(2, –3)	
(3, 1)	
(3, 2)	
(3, 3)	
(3, –1)	
(3, –2)	
(3, –3)	
(–1, 1)	
(–1, 2)	
(–1, 3)	
(–1, –1)	
(–1, –2)	
(–1, –3)	
(–2, 1)	
(–2, 2)	
(–2, 3)	
(–2, –1)	
(–2, –2)	
(–2, –3)	
(–3, 1)	
(–3, 2)	
(–3, 3)	
(–3, –1)	
(–3, –2)	
(–3, –3)	



Once you have the slope field drawn, locate the point representing the initial condition (0, 3) and draw an approximate graph of the solution to the differential equation. Then locate the point representing the initial condition (0, –3) and draw this solution. Your graph should be approximately circular.

Now use *Converge* to graph the slope field and the two solutions:
Choose **C**alculus, then **G**raph solutions of 1st order diff. eq..
Type: **–x/y <Enter>**
Use this window: Min X: **–4** Max X: **4**, Min Y: **–4** Max Y: **4** Then hit **<F5>**.

Compare the slope field to the one you drew.

Then type in the initial condition: x = **0**, y = **3** and compare the solution to your graph.

Finally, type in the initial condition x = **0**, y = **–3** and compare the solution to your graph.

Algebraically, we will again use separation of variables. This time you solve the differential equation. (Hint: multiply both sides of the equation by *y dx*.

$$\frac{dy}{dx} = -\frac{x}{y}$$

TAYLOR POLYNOMIALS

Example 1: Find the first ten Taylor polynomials for $f(x) = \sin x$ at $x = 0$.

You should have completed all the calculations for these polynomials before starting this supplement and these are the polynomials you should have:

$$p_1(x) = p_2(x) = x$$

$$p_3(x) = p_4(x) = x - \frac{x^3}{3!}$$

$$p_5(x) = p_6(x) = x - \frac{x^3}{3} + \frac{x^5}{5!}$$

$$p_7(x) = p_8(x) = x - \frac{x^3}{3} + \frac{x^5}{5!} - \frac{x^7}{7!}$$

$$p_9(x) = p_{10}(x) = x - \frac{x^3}{3} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!}$$

Now we will use *Derive* to compute and graph these Taylor polynomials. From our results above, we will get only the odd-numbered polynomials.

In *Derive*,

Author, **sinx <Enter>**	The function $\sin(x)$ is now on line #1.
Calculus, **T**aylor, **<Enter>, <Enter>, 1 <Ctrl> <Enter>**	This is the command for the first degree Taylor polynomial at $x = 0$. Using <Ctrl><Enter> eliminates the need to Simplify the resulting expression. You should have x on line #2.
Calculus, **T**aylor, **1 <Enter>, <Enter>**	
3 <Ctrl><Enter>	You should have $x - \frac{x^3}{6}$ on line #3.

Continue to get the remaining polynomials up through degree 9. Then hit the **<Home>** key to highlight sin(x). We will now plot sin(x) and the Taylor polynomials.

Plot, **B**eside, **<Enter>**	This opens up a graphing window and switches to it.
Range, **–2<Alt>p <Tab> 2<Alt>p <Tab> –3 <Tab> 3 <Enter>**	This changes the graphing window.
Plot	The graph of $f(x) = \sin(x)$ is in window 2.
<F1>, ↓, **<F1> P**lot	The graph of $p_1(x)$ is superimposed.
<F1>, ↓, **<F1> P**lot	The graph of $p_3(x)$ is superimposed.
<F1>, ↓, **<F1> P**lot	The graph of $p_5(x)$ is superimposed.
<F1>, ↓, **<F1> P**lot	The graph of $p_7(x)$ is superimposed.
<F1>, ↓, **<F1> P**lot	The graph of $p_9(x)$ is superimposed.

You should notice that as the degree of the polynomial increases, the polynomial better approximates sin(x) for a larger interval centered at $x = 0$.

Example 2: Superimpose the graph of the degree 49 Taylor polynomial of $f(x) = \sin x$ on the graphs you did in Example 1.

<F1>, **C**alculus, **T**aylor, **1 <Enter>, <Enter>, 49, <Ctrl><Enter>**
<F1> Plot
These graphs provide striking evidence that as the degree of the Taylor polynomial gets larger and larger, the polynomial provides a better and better approximation to the function.

Example 3: Graph the function $f(x) = \arctan x$ and a high-degree (31) Taylor polynomial to see why the interval of convergence is $-1 < x < 1$.

1. In Window 1, **R**emove, **1 <Enter>**
2. **A**uthor atanx
3. **<F1>**
4. **D**elete, **A**ll
5. **R**ange: **–1.8 <Tab> 1.8 <Tab> –3 <Tab> 3 <Enter>**
6. **P**lot
7. **<F1>**
8. **C**alculus, **T**aylor, **<Enter>, <Enter>, 31, <Ctrl> <Enter>**
9. **<F1> P**lot

Here is striking visual support that the interval of convergence is only between –1 and 1.

TAYLOR S THEOREM

Example: Use the Maclaurin polynomial of degree 7 to approximate $\arctan(0.3)$ and determine the decimal accuracy.

Use *Derive* to **A**uthor **atan x** and compute the first seven derivatives of arctan *x.* Record them here:

$f(x) = \arctan x$

$f'(x) =$

$f''(x) =$

$f'''(x) =$

$f^{(4)}(x) =$

$f^{(5)}(x) =$

$f^{(6)}(x) =$

$f^{(7)}(x) =$

Now evaluate each of these derivatives at $x = 0$, using the **M**anage, **S**ubstitute command. (**M**anage, **S**ubstitute, type line # and hit **<Enter>,** type **0** and hit **<Ctrl><Enter>**) Record your results here:

$f(0) =$

$f'(0) =$

$f''(0) =$

$f'''(0) =$

$f^{(4)}(0) =$

$f^{(5)}(0) =$

$f^{(6)}(0) =$

$f^{(7)}(0) =$

Next, since the 7^{th} degree Maclaurin polynomial about $x = 0$ is given by

$$p_7(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \ldots + \frac{f^{(7)}(0)}{7!}x^7$$

we can now write the 7^{th} degree Taylor polynomial for arctan x at $x = 0$:

$$p_7(x) =$$

Use *Derive* to check this answer:

Remove, **2** <**Enter**>	The only expression remaining should be atan(x) on line #1.
Calculus, **T**aylor, **<Enter>, <Enter>** **7 <Ctrl><Enter>**	This should agree with your $p_7(x)$.

We will now plot both of these:

<Home>, Plot, **B**eside, **<Enter>**	This opens up a graphing window and switches to it.
Plot	This graphs arctan x.
<F1>, ↓, <F1>, Plot	This graphs $p_7(x)$.

Notice the close agreement between arctan x and $p_7(x)$ near x = 0.

Now use $p_7(x)$ to approximate arctan(0.3):

<F1>, Manage, **S**ubstitute, **<Enter>,**
0.3 <Ctrl><Enter>

appro**X**imate, **<Enter>**

Thus, arctan 0.3 $\approx$ 0.291454. Now we need to see how accurate this answer is.

Taylor's Theorem lets us determine an upper bound on the error involved when $p_n(b)$ is used to approximate $f(b)$. If $p_n(x)$ is the n^{th} degree Taylor polynomial at $x = a$ and $p_n(b)$ is used to approximate $f(b)$, then $R_n(b) = \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}$, where c is some number between a and b. In this case, a = 0, b = 0.3, and n = 7, so $R_7(0.3) = \frac{f^{(8)}(c)}{8!}(0.3)^8$, where c is a number between 0 and 0.3. Since we do not know c, we cannot find $R_7(0.3)$ but we can find a maximum value for $|R_7(0.3)|$ by using the largest possible vlaue of $f^{(8)}(x)$ on the interval (0, 0.3) as an upper bound on $f^{(8)}(c)$. We can determine a suitable upper bound by graphing $f^{(8)}(x)$ with *Derive*.

In Window 1,

Calculus, **D**ifferentiate, **1 <Enter>, <Enter> 8 <Ctrl><Enter>**	This gives $f^{(8)}(x)$.
Author, **abs <F4> <Enter>**	This gives $\left|f^{(8)}(x)\right|$.
<F1> Delete, **A**ll	This clears the graphing window.
Range, **–.1** (clear digits) **<Tab> .5** (clear digits) **<Enter>**	This sets the x endpoints of the graphing window.
Scale, **<Tab> <Tab>, Y**es **<Enter>**	This selects automatic scaling for the y-axis.
Plot	You should have the graph of $\left|f^{(8)}(x)\right|$

<F3> This turns on the trace feature.

Now use ← or → to trace to the maximum value of $\left|f^{(8)}(x)\right|$ on the interval (0, 0.3). If you zoom in on the obvious maximum point, you will probably agree that a suitable upper bound is 4392. We now have

$$\left|R_7(0.3)\right| = \left|\frac{f^{(8)}(c)\,(0.3)^8}{8!}\right| < \frac{(4392)(0.3)^8}{8!}$$

Use *Derive* to evaluate this expression:

In Window 1, **A**uthor, **(4392)(.3)^8 / 8! <Enter>**

approo**X**imate, **<Enter>**

We see that an upper bound for the absolute value of the error 7.14680×10^{-6} or 0.00000714680, which means 4-place decimal accuracy.

Thus, arctan(0.3) = 0.291454 with 4-place decimal accuracy.

SEQUENCES AND SERIES USING DERIVE

For all these examples, Approximate Precision is best.

I. SEQUENCES

Example 1: A convergent sequence : $\left\{ \frac{n+1}{n} \right\}$

Generate the first 50 terms of the sequence:

1. **A**uthor **(n + 1) / n**
2. **A**uthor: **vector ([n , <F3>] , n , 1 , 50)**

   When typing, take care with parentheses, brackets, and commas. Note that <F3> brings down the highlighted expression.

3. **S**implify, **<Enter>**

   You have now generated a list of ordered pairs $(n,\ f(n))$ for $1 \le n \le 50$. You can inspect the list by hitting $\rightarrow$ repeatedly. When you are finished inspecting this list, press the $\uparrow$ to highlight the entire list.

Copy a representative sampling of the terms of the sequence onto this table:

n	***f*(*n*)**

We will now graph this set of ordered pairs.

4. **P**lot, **O**verlay
5. **R**ange, **0 <Tab> 50 <Tab> 0 <Tab> 2 <Enter>**
6. **P**lot

   You now have a graph of the sequence which clearly shows how the sequence converges to 1.

Copy the graph here:

Example 2: A sequence of alternating positive and negative terms: $\left\{\left(-\frac{2}{3}\right)^n\right\}$

1. **A**uthor **(– 2 / 3) ^ n <Enter>**
2. **A**uthor: **vector ([n , <F3>] , n , 1 , 50) <Enter>**
3. **S**implify, **<Enter>**

Inspect by →. When finished hit ↑.

Copy a sampling of the list here:

n	***f*(*n*)**

4. **P**lot
5. **D**elete, **A**ll
6. **R**ange, 0 <Tab> 50 <Tab> –1 <Tab> 1 <Enter>
7. **P**lot

Here we see a sequence of alternating positive and negative terms converging to 0.

Copy the graph here:

II. INFINITE SERIES

We now move on to infinite series. We define a series as convergent if its sequence of partial sums converges. However, computing the terms of the sequence of partial sums is a tedious job by hand. *Derive* gives us the power to compute many terms of the sequence of partial sums in order to get some idea of the convergence or divergence of the series. Here are some examples.

Example 1: A convergent geometric series: $\sum_{n=0}^{\infty} \frac{1}{2^n}$

We will generate the first 51 terms of the sequence of partial sums for this geometric series.

1. **A**uthor: **1 / 2 ^ n <Enter>**
2. **C**alculus, **S**um, **<Enter>, <Enter>, 0, <Tab>, k <Enter>**
3. **A**uthor: **vector ([k , <F3>] , k , 0 , 50) <Enter>**
4. **S**implify. **<Enter>**

Inspect by →**.** When finished hit ↑**.**

Copy a sampling of this sequence of partial sums onto this table:

k	$\sum_{n=1}^{k} \frac{1}{2^n}$

Now obtain a graph of the sequence of partial sums.

5. **P**lot
6. **D**elete, **A**ll
7. **R**ange: **0 <Tab> 50 <Tab> 0 <Tab> 3 <Enter>**
8. **P**lot

Here we see a convergent sequence of partial sums indicating an infinite series whose sum is 2. Copy the graph here:

Example 2: A divergent series: $\sum_{n=1}^{\infty} \frac{3n}{3n+1}$

1. **A**uthor: **3n / (3n + 1) <Enter>**
2. **C**alculus, **S**um, **<Enter>, <Enter>, 1, <Tab>, k <Enter>**
3. **A**uthor: **vector ([k , <F3>] , k , 1 , 50) <Enter>**
4. **S**implify, **<Enter>**

Inspect by $\rightarrow$**.** When finished hit $\uparrow$.

Copy a sampling of this sequence of partial sums onto this table:

k	$\sum_{n=1}^{k} \frac{3n}{3n+1}$

Now obtain a graph of the sequence of partial sums.

5. **P**lot
6. **D**elete, **A**ll
7. **R**ange: 0 <Tab> 50 <Tab> 0 <Tab> 60
8. **P**lot

Here we see a divergent sequence of partial sums indicating a divergent series.
Copy the graph here:

Example 3: P-Series: $\sum_{n=1}^{\infty}\frac{1}{n^3}$, $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$, and $\sum_{n=1}^{\infty}\frac{1}{n}$.

1. **A**uthor: **1 / n ^ 3 <Enter>**
2. **C**alculus, **S**um, **<Enter>, <Enter>, 1, <Tab>, k <Enter>**
3. **A**uthor: **vector ([k , <F3>] , k , 1 , 50) <Enter>**
4. **S**implify, **<Enter>**

Inspect by →. When finished hit ↑**.**

5. **P**lot
6. **D**elete, **A**ll
7. **R**ange: **0 <Tab> 50 <Tab> 0 <Tab> 4 <Enter>**
8. **P**lot

Inspecting the list and the graph, we see that this series converges to approximately 1.2.

9. **A**uthor: **1 / <alt> q n <Enter>**
10. **C**alculus, **S**um, **<Enter>, <Enter>, 1, <Tab>, k <Enter>**
11. **A**uthor: **vector ([k , <F3>] , k , 1 , 50) <Enter.**
12. **S**implify, **<Enter>**

Inspect by →. When finished hit ↑.

13. **P**lot
14. **D**elete, **A**ll
15. **R**ange: **0 <Tab> 50 <Tab> 0 <Tab> 15 <Enter>**
16. **P**lot

Here we can see a divergent sequence of partial sums.

Now, the harmonic series. Since the harmonic series diverges so slowly, we will need to compute 200 terms of the sequence of partial sums.

17. **A**uthor: **1 / n <Enter>**
18. **C**alculus, **S**um, **<Enter>, <Enter>, 1, <Tab>, k <Enter>**
19. **A**uthor: **vector ([k , <F3>] , k , 1 , 50) <Enter>**
20. **S**implify, **<Enter>**

Inspect by →. When finished hit ↑.

21. **P**lot
22. **D**elete, **A**ll
23. **R**ange: **0 <Tab> 200 <Tab> 0 <Tab> 6 <Enter>**
24. **P**lot

Example 4: Alternating Series: $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$ and $\sum_{n=0}^{\infty} (-1.05)^n$.

Alternating series are particularly interesting to graph, since you can clearly see the sequences of odd and even partial sums decreasing or increasing.

1. **A**uthor **(– 1) ^ (n + 1) / <alt> q n <Enter>**
2. **C**alculus, **S**um, **<Enter>, <Enter>, 1, <Tab>, k <Enter>**
3. **A**uthor, **vector ([k , <F3>] , k , 1 , 50) <Enter>**
4. **S**implify, **<Enter>**

Inspect by →. When finished hit ↑.

5. **P**lot
6. /**D**elete, **A**ll
7. **R**ange: **0 <Tab> 50 <Tab> 0 <Tab> 1.2 <Enter>**
8. **P**lot

Here we can clearly see the two sequences converging to the sum.

9. **A**uthor **(– 1.05) ^ n <Enter>**
10. **C**alculus, **S**um, **<Enter>, <Enter>, 0, <Tab>, k <Enter>**
11. **A**uthor, **vector ([k , <F3>] , k , 0 , 50) <Enter>**
12. **S**implify, **<Enter>**

Inspect by →. When finished hit ↑.

13. **P**lot
14. **D**elete, **A**ll
15. **R**ange: **0 <Tab> 50 <Tab> –6 <Tab> 6 <Enter>**
16. **P**lot

This graph clearly shows a divergent sequence of partial sums.

APPENDIX: INTRODUCTION TO DERIVE

The DERIVE Screen

Once you have selected *Derive*, the initial *Derive* screen is displayed on your monitor. Notice the two-line **command menu** at the bottom of the screen. There are two ways to choose an option from the command menu: You can type the capitalized letter in the option, or you can press the Space bar until the desired option is highlighted and then press <enter>. For example, to choose Author you would type A, to choose soLve, you would type L.

Try This:

In window 1, type A (for Author), then type x^2−5x−4 and hit <enter>

Notice that the expression $x^2 - 5x - 4$ appears in standard mathematical format in window 1.

Graphing

There are two ways to obtain a graph of this function with *Derive,* using a full-screen plotting window, and splitting the screen so that the graphing window is next to (or under) the algebra window. We will first plot the graph with a full-screen window. To do this:

Type P (for Plot), type O (for Overlay). Then type P (for Plot).

However, the complete graph does not show because of the window dimensions.

Window Dimensions

When the coordinate system is first set up, the scale on the x and y axes is 1, so the window is $-4 \le x \le 4$ and $-3 \le y \le 3$. With this window, each tic mark on the x and y axes represents 1 unit. You can read the scale on the bottom of the screen just to the right of center. Often this window needs to be changed and this can be done in two ways: with the Scale command and with the Range command. We will first use the Scale command to change the scale so that each tic mark on the x-axis represents 2 units and each tic mark on the y-axis represents 5 units.

Type S (for Scale), type 2 <Tab> 5 <enter>

The graph is immediately redrawn in the new scale. Notice that the new scale appears at the bottom of the screen.

An alternate method of changing the window size is the Range command. This command sets the left, right, bottom, and top boundaries of the window. We will now use the Range command to get a window of $-6 \le x \le 10$ and $-15 \le y \le 30$.

Type R (for Range), type −6 (hit the spacebar to clear out the extra digits) **<Tab> 10** (again use the spacebar to clear out extra digits), **<Tab> −15 <Tab> 30. Then hit <Enter>.**

Once again the graph is redrawn using the new window size.

Now return to the algebra window as follows:

Type A (for Algebra).

Splitting the Screen

In many applications, it is desirable to have the algebraic expressions on one side of the screen and the graphs on the other side. With *Derive* we can split the screen into two windows, one for algebra and one for graphs.

Do this now:

Type W (for Window), S (for Split), V (for Vertical), <enter>

The **<F1>** function key allows you to toggle back and forth between the two windows. Try it a few times and stop at window 2. We will now tell *Derive* that window 2 will be the graphics window:
In window 2, type W (for Window), D (for Designate), 2 (for 2d-plot). Type Y (to abandon expressions).
Derive is now ready to perform calculations in window 1 and graph functions in window 2.
Graph the quadratic function $y = x^2 - 5x - 4$ as follows:
In window 2, change the window using either Scale or Range as you did before, then type P (for Plot)
The function is graphed in the plotting window.

Moving the Cross-Hair
In the graphing window, you will notice a small cross-hair. The current coordinates of this cross-hair can be read on the bottom left of the screen. The criss-hair can be moved about the coordinate system in two ways:
To move the cross-hair right or left: Press **<Ctrl>** $\rightarrow$ or **<Ctrl>** $\leftarrow$ for larger moves or press just $\rightarrow$ or $\leftarrow$ for very small moves.
To move the cross-hair up or down: Press **<PgUp>** or **<PgDn>** for larger moves or $\uparrow$ or $\downarrow$ for very small moves.
These movements are useful for locating points of intersection, zeros, relative extrema, etc.
Alternatively, the cross-hair can be moved to a specific location as follows:
Type M (for Move), type the x-coordinate of the desired location, hit the <Tab> key, type the y-coordinate <enter>
Now use either of these two methods to move the cross-hair to the *x*-intercepts of the graph and read the coordinates off the bottom of the screen. You should get approximately –0.6944 and 5.6944.
To determine the exact value of these *x*-intercepts, we need to return to the algebra window and learn about precision and the soLve command.

Precision
When *Derive* is first started, it is in **exact** precision. This means that it will compute exact values (as opposed to decimal approximations) whenever possible. There are other alternatives.
In window 1, type O (for Options), P (for Precision)
Notice that Exact is highlighted but the other two options are Approximate and Mixed. Mixed precision means that *Derive* will look for an exact solution first and if it cannot find one, it will look for an approximate solution. Approximate means that *Derive* will use numerical techniques to approximate a solution but you must tell *Derive* where to look by specifying an *x*-interval.
Leave *Derive* in exact precision by pressing <**Esc>** twice. Now find the exact values of the *x*-intercepts by solving the equation $x^2 - 5x - 4 = 0$ as follows:
Type L (for soLve), and hit <enter> to accept expression #1.
NOTE: When you use the soLve command, *Derive* assumes that you are setting the expression equal to zero.
Notice that you get the exact (radical) values of the solutions. You can see how well the approximate values you got using the cross-hair agree with these exact values by getting the decimal values of the radical expressions as follows:
Hit the $\uparrow$ to highlight the first solution and then type X (for approXimate) and hit <enter>.

You should get 5.70156. The value from the cross-hair was fairly close.
Now hit the ↑ to highlight the second solution and again type X (for apprXimate) and hit <enter>.
You should get –0.701561.

New Problem
Solve $\sin x - x^2 = 0$. To begin, we should remove all expressions from window 1 and delete the graph in window 2.

Removing Expressions
In the algebra window, it is often useful to remove some or all of the expressions that have piled up. This can be done as follows:
Type R (for Remove), type 1 (to begin removal with line #1) and then hit <enter>.

Deleting Graphs
You can remove graphs from the display in two ways:
Type D (for Delete), then choose one of the options: A (for All), L (for Last), B (for all But last), F (for First).
When you use the Delete option, all graphs not deleted will remain and the window will remain unchanged.
As an alternative, you can delete all graphs and return the window to its original status by:
Type W (for Window), D (for Designate), 2 (for 2D-plot)
For this problem use this last method to remove the graph.
Now to solve the problem of solving the equation $\sin x - x^2 = 0$.
In window 1, type A (for Author), then type sinx–x^2 <enter>
Hit <F1> and type P (for Plot).
Notice from the graph that there appear to be two solutions. One solution is $x = 0$ and this clearly checks in the equation. The other solution is between 0 and 1 and by moving the cross-hair, you will find that the value is approximately 0.8. To find this value algebraically:
Hit <F1> to return to window 1, type L (for soLve) and hit <enter>
Remember that you are in exact precision so *Derive* **only** looks for exact solutions. Since there are no exact solutions to this equation, *Derive* returns the equation unsolved. Since the graph clearly shows a solution, we will find an approximate solution as follows:
Type O (for Options), P (for Precision), A (for Approximate), and <enter> to accept 6 significant digits.
Then type L(for soLve) and hit <enter>
Derive now asks for a lower and upper bound on the solution. Since we saw from the graph that the solution is about 0.8, we could use 0 as the lower bound and 1 as the upper bound:
Type 0, hit the <Tab> key, then type 1, hit <enter>.
Notice that *Derive* returns the solution $x = 0$, which is not the solution we were looking for! *Derive* assumes there is only one solution in the interval you specify and stops when it finds it. We need to specify an interval that does not include 0, so try the interval (0.5,1):
Type L, type 2, type .5 <Tab> 1, <enter>
This time we are successful and get the value 0.876726.

Typing Expressions
Naturally, it is important to type expressions correctly and use parentheses whenever necessary.

Suppose we author the expression $\frac{3x^2-2x+1}{5x-4}$ **incorrectly** as follows:

Type A, type 3x^2–2x+1/5x-4 <enter>.

Notice that without parentheses the expression is $3x^2-2x+\frac{1}{5}x-4$. We need to correct this as follows:

Type A, then hit the <F3> key.

The <F3> key will bring whatever expression is highlighted down to the authoring line. (Note: The <F4> key will bring the highlighted expression down enclosed in parentheses). Now we need to edit this expression by inserting two sets of parentheses.

Try moving the cursor by hitting the ← key. Notice that nothing happens on the authoring line but watch the highlighted expression. The arrow keys are used to change the parts of the expression that are highlighted. Now, only x^2 is highlighted. Press the → to highlight 3*x* then press the ↓ key. Now only the 3 is highlighted. This ability to isolate parts of an expression can be very useful when trying to solve equations. However, it does not help us insert the parentheses we need. To do this we need to change the role of the arrow keys so that they apply to the authoring line. To do this we use the <F6> key.

Hit <F6> (you will notice that Lin shows up on the bottom of the screen to indicate that line editing can take place), **then hit the <Home> key to get to the beginning of the expression, hit the <Ins> key to get into insert mode, type (, use the → to get to the end of the numerator, type) , use the → to get to the beginning of the denominator, type (, hit the <End> key, and type the final). Hit <enter>.**

The expression now appears correctly.

Evaluating Functions

There are often times when you need to evaluate an expression or a function at one or more values of the independent variable. There are two ways to do this: by using the **M**anage, **S**ubstitute command and by defining the expression as a function.

The Manage, Substitute Command

Suppose you want to evaluate the expression $\frac{4.2x^2-3.5}{11.6x^2+2.7}$ for x = –2, –1, and 3. First, remove all the expressions in the algebra window as you did before. Then close window 2 as follows:

In window 2, type W (for Window), C (for Close), and hit <Enter> to accept window 2.

Now author the expression:

Type A (for Author), type (4.2x^2–3.5)/(11.6x^2+2.7) and hit <Enter>.

Now we will use the Manage, Substitute Command to evaluate the expression at $x = -2$ as follows:

Type M (for Manage), type S (for Substitute), hit <Enter> to accept expression #1.

At this point the menu line reads: MANAGE SUBSTITUTE value: and there is an x with a blinking cursor under it. Under this line a message reads "Enter replacement for x". At this point you only need to type in the value –2, so

Type –2 and hit <Enter>.

What you see on line #2 is the expression with x replaced by –2. In order to evaluate the expression, you need to use the **S**implify command.

Type S (for Simplify) and hit <Enter> to accept line #2.
Notice that the value is in decimal form. That's because we put *Derive* into Approximate precision. If you want the fraction form, you should be in Exact precision.
Now, evaluate the expression for $x = -1$.
Type M (for Manage), S (for Substitute), type 1 (for expression #1), hit <Enter>, type −1 and hit <Enter>. Type S (for Simplify) and hit <Enter>.
You can shorten this process and eliminate the need to use the **S**implify command as follows: once you have typed in the value for *x*, do not hit <Enter>. Instead, hit and hold the <Ctrl> key and press <Enter>. This results in getting the final decimal value. Do this to find the value of the expression at $x = 3$.
Type M, S, type 1, hit <Enter>, type 3 and hit <Ctrl><Enter>.
An alternate method for evaluating expressions is to write the expression as a function.

Functions
We usually write a function as $f(x)$. In order for *Derive* to recognize this collection of letters and symbols as meaning a function, you need to use a special notation. Type the name of the function, such as $f(x)$ and then type a colon followed by an =. Suppose we want to define the expression we were just working with as $f(x)$. It should be on line 1, so use the ↑ key to highlight expression #1.
Now type A (for Author), type f(x):= and then hit the <F3> key to bring down expression #1 and hit <Enter>.
Now that *Derive* knows this expression is a function, we can find $f(-2), f(-1)$, and $f(3)$:
Type A (for Author), type f(−2), hit <Enter>. Type S (for Simplify), and hit <Enter>.
The result is the same value we got before. As you may have guessed, you can omit the **S**implify command by using <Ctrl><Enter> after you type f(−2). Try this to find $f(-1)$ and $f(3)$:
Type A, type f(-1), hit <Ctrl><Enter>.
Type A, type f(3), hit <Ctrl><Enter>.
Now you know two ways to evaluate expressions. In general, **M**anage, **S**ubstitute is easier when you need only one or two values, but defining a function is faster if you need many values.

This introduction to *Derive* should be enough to get you started but it barely begins to illustrate the many things *Derive* can do to help you with your mathematics. Experiment, try other commands, like **F**actor and **E**xpand and see what they do. *Derive* is a very useful and important tool and the more familiar you are with it, the more useful it will be to you.