

Beginning Algebra Students' Images of the Function Concept

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AMATYC's *Crossroads In Mathematics* lists function as one of the Standards of Content. This paper describes research focussed on the feasibility of using the function concept as a core idea in developmental mathematics. This research considers different aspects that make up the function concept, building a view of function along both breadth (**facets** or representations) and depth (**layers**) dimensions. Pre- and post-course surveys along with task-based interviews are used to build a profile of developmental algebra students' concept image of function.

Introduction

U.S. college mathematics departments today are faced with a sizable percentage of the student body that must begin their college career in a non-credit mathematics course. Courses that fall under this umbrella include arithmetic, geometry, beginning algebra, and intermediate algebra. In 1990, 58% of the 1.3 million community college students taking mathematics were enrolled in Intermediate Algebra or below (Watkins et al., 1993). Many developmental algebra students have been severely debilitated by their previous exposure to mathematics, including algebra. Succeeding with this population may require providing the students with an educational experience that is markedly different from that experienced previously. One option makes function the core concept. A focus on function in developmental algebra might be a viable alternative to the standard skills-based developmental algebra courses.

The research discussed in this paper focuses on the understanding of functions that students acquire as a result of completing a "reform" beginning algebra curriculum in which "function" is the key concept throughout. The important question is: Can adult students who arrive at college having had debilitating prior experiences with algebra develop a rich concept image of "function" through appropriate instructional treatment? Students at four community colleges completed written pre- and post-course surveys questioning their understanding of functions. Analysis of this data suggests statistically significant positive shifts in students' ability to answer questions involving equations in two variables, two-column tables, graphs, and written definitions. Several of these students have undergone interviews designed to probe, in depth, their understanding of function. Based on this data, profiles that visually depict students' concept image of function along both breadth (facets) and depth (layers) dimensions are being developed. Of particular interest is students' ability to demonstrate cognitive connections across various facets. Some results are discussed in this paper.

Creating Debilitated Students

Critical to understanding the nature of the debilitation that has occurred in beginning algebra students is the concept of procept, introduced by Gray and Tall (1994). Gray and Tall write: “An **elementary procept** is the amalgam of three components: a *process* that produces a mathematical *object*, and a *symbol* that represents either the process or the object” (p. 121). For example, the notation $f(x)$ where f is the name of a function contains the ideas of a process to follow to produce output from input, an object—a function, and a symbolism $f(x)$ that can represent either the process or the object. The ambiguity of the symbolism gives the symbolism its power. Those who can move easily between the various meanings of such symbolism have compressed the information inherent in the symbolism. Meaning for symbols often develops by first doing **procedures** such as evaluating a function at a given number. Procedures may then mature into **processes** in which the idea that a function produces output from input is understood by the student without having to apply a specific algorithm to an input to get an output. At some point, the concept may mature in the student’s mind to the point where it can be thought about as an **object**. In this case, the student can perform operations on this object “function,” such as composition or differentiation. The ability to think flexibly about a concept, such as function, as both a process and an object is referred to as **proceptual thinking**. Contrasting this, thinking that is dependent on the selection and performance of appropriate procedures is called **procedural thinking**. Tall (1996) writes: “Procedures allow individuals to do mathematics, but learning lots of separate procedures and selecting the appropriate one for a given purpose becomes increasingly burdensome. Procepts allow the individual not only to carry out procedures, but to regard symbols as mental objects, so they can not only do mathematics, they can think about the concepts” (p. 12). Students who rely on procedural thinking are doing much more difficult mathematics. Mathematics, for them, consists of disconnected cognitive units of algorithms triggered by a specific problem format. The divergence between those who interpret processes only as procedures and those who see them as flexible procepts is called the **proceptual divide** (Gray & Tall, 1991). In essence, the more able depend on procepts while the less able depend on procedures. As cognitive strain grows, a student, who up to that point may have been successful, encounters difficulty and asks “tell me how to do it,” desiring the security of a procedure rather than the flexibility of a procept. From this point on failure is almost inevitable. If someone who has turned to procedural thinking finds security, then additional practice of those procedures does little more than widen the proceptual divide. Developmental algebra instructors regularly encounter symptoms of the proceptual divide. Their students try to memorize so many procedures with so little understanding that algebra is a mishmash of disconnected procedures. For example, when confronted with an algebraic expression such as $2x + 3$, many will set the expression equal to zero and solve for x . Students feel they must perform a procedure when given an algebraic expression. As Tall states, “... the difficulties that average college students have with algebra occur because of previous rule-bound approaches to the subject. When students do not understand what something is, at least they can get temporary success by becoming secure with procedures to do things with it” (1992, p. 3).

Focussing on Function

The fact that many students in beginning algebra have had algebra before results in a mental network of concept images rife with misinformation. Such cognitive networks contain **concept images** that conflict with **concept definitions** (Tall & Vinner, 1981). Concept definition refers to the mathematical definition of a concept, while concept image is everything associated in somebody's mind with the concept name. The students' prior exposure to the concepts is detrimental due to the inappropriate existing network. As Ausubel (1968) stated, "The most important single factor influencing learning is what the learner already knows" (p. vi). It is often the case that what the learner knows is replete with misconceptions. Personal observation suggests that students' concept images of variables and equations displays little overlap with the definition of these concepts, for example. What happens when a new idea is presented in this context? Consider what Hiebert & Carpenter (1992) say about existing networks: "If the learner tries hard to fit a new idea, fact, or procedure into a current way of thinking, existing networks constrain the relationships that are created. At the other extreme, a learner may represent new information in a way that does not connect it with existing networks" (p. 70). Two potential problems arise. An existing network that is incorrectly constructed constrains the construction of the web for a new idea. The result is a construction rife with misconceptions. On the other hand, if the new idea does not connect with the existing networks, a lack of connections between old and new occurs leading to disjointed and unusable knowledge.

Due to prior acquaintance with numerous concepts, students exhibit profound learning interferences. Students are often adept at solving linear equations procedurally, and, as a result, resist considering the meaning of such equations or how they arise in various situations. They know how to "solve it," so what's the point in learning more about the concept? As stated in Simoneaux & Kirshner,

... the experience of memorizing material and being able to regurgitate it equated to learning for these students. Memorization inhibits free, open-ended, creative explorations of ideas and materials...Rote learning sets up superficial associations related to solution procedures. These may conflict with subsequent meaningful instruction. In such cases, either prior structures remain, thus making new relationships impossible; or, structures have to be unlearned and new relationships constructed. This unlearning and relearning creates unnecessary obstacles (interferences). Thus when initial mathematics of a concept focuses on memorizing procedures, facts, and definitions, subsequent meaningful learning may be impaired. (1994, pp. 222–223)

In light of the hindrances placed on students by previous exposure to algebra, one approach to teaching developmental algebra suggests that the course must be radically different from the one students originally took. One option makes function the core concept, following the philosophy of Yerushalmy and Schwartz, who state: "... we believe that function is the fundamental object of algebra and that it ought to be present in a variety of representations in algebra teaching and learning from the outset" (1993, p. 41). AMATYC, in its Standards document (1995), recommends that "function" be one of the central themes of the Standards for Content (p. 13).

However, there are those who worry that introducing function at this level may introduce new obstacles to students. Sierpiska (1992), for example, writes: "Lack of algebraic awareness makes the understanding of function very difficult if not impossible" (p. 45). This proposal suggests that we use function as a vehicle to develop algebraic awareness.

"Reform" Curriculum

Students involved in the study were enrolled in "pilot" sections of beginning algebra at 4 community colleges. The key "reform" component of the pilot sections was the text. *Mathematical Investigations: An Introduction to Algebraic Thinking* (DeMarois, McGowen, & Whitkanack, 1996) focuses on student investigation of problems based on a pedagogical approach that uses a constructivist theoretical perspective of how mathematics is learned (Davis et al., 1990). The authors subscribe to the theoretical perspective that the main concern in mathematics should be "with the students' construction of schemas for understanding concepts. Instruction should be dedicated to inducing students to make these constructions and helping them along in the process" (Dubinsky, 1991, p. 119). Each unit begins with an investigation of a problem situation. Following the gathering of data, students work collaboratively on tasks based on the investigation activities. A discussion in the text summarizes essential mathematical ideas. The instructor orchestrates inter-group and class discussions of the investigations. Explorations are assigned to broaden and deepen the knowledge students are expected to have constructed during successive steps of the cycle.

The materials focus on development of mathematical ideas using a core concept of function. Function is initially defined as "a process that receives input and returns a unique value for output" (DeMarois, McGowen, & Whitkanack, p. 92). Each function is based in a problem situation. Functions are investigated numerically, graphically, and with function machines before the symbolic form is created. Tables, equations, graphs, function machines, and verbal and written descriptions are all used to analyze relationships.

Research Question and Model for Analysis

As noted in the Introduction, the driving question of this research is: Can adult students who arrive at college having had debilitating prior experiences with algebra develop a rich concept image of "function" through appropriate instructional treatment? To test this question, both quantitative and qualitative data is collected to develop a profile of students' concept images of function. The profiles include both a measure of breadth of understanding and depth of understanding. This research builds on the development described by Schwingendorf et al. (1992) who contrast the vertical development of the concept implying increasing depth and the horizontal development relating different representations and implying breadth.

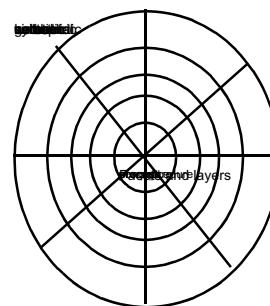
The breadth dimension is conceived as consisting of various representations, including geometric, numeric, and symbolic. DeMarois & Tall (1996) introduce the word **facet** to build up a description of the breadth dimension. Webster's *New World Dictionary* (Guralnik, 1980 p. 300) defines a facet as "any of a

number of sides or aspects.” The facets of a mathematical entity refer to various ways of thinking about it and communicating to others, including verbal (spoken), written, kinesthetic (enactive), colloquial (informal or idiomatic), notational, numeric, symbolic, and geometric (visual) aspects. These are not intended to be independent or exhaustive, but provide a suitably broad framework to begin an analysis of the function concept.

An area that has received much attention is students' ability to move comfortably between facets. This implies that they can choose the most appropriate facet to use for a given problem. Cuoco (1994, p. 125) suggests that the connections between “representations” are properties of a “higher-order function.” A linking of the numeric, symbolic, and geometric facets is briefly discussed in this paper. However, this is an area that is the subject of on-going research.

DeMarois & Tall (1996) use the term **layer** to refer to various levels of the depth dimension. Webster's Dictionary describes a “layer” as “a single thickness, coat, or stratum.” In this framework, pre-procedure, procedure, process, concept, and procept are considered layers of increasing depth. Pre-procedure assumes that the student is on the ground floor, so to speak, with respect to a concept. A procedure is a coherent sequence of actions—a schema of actions. A procedure is a “specific algorithm,” as mentioned earlier. On the other hand, a process is a cognitive entity, not dependent on individual steps, but rather on the result produced from the original input. For example, the expressions $2x + 6$ and $2(x + 3)$ represent two different procedures. The results of applying each procedure to a given input are the same. Students who view these as different functions might be classified at the procedure layer while those who classify these as the same function might be placed at the process layer. Students at the procedure layer are dependent on the procedure performed to obtain output from input. Students at the process layer can cognitively accept the existence of a process between input and output without needing to know the specific algorithm. The concept layer aligns closely with the ability to treat the mathematical idea as an object to which a procedure can be applied. After the concept layer, a procept layer is placed, to indicate the flexibility to move easily between process and object layers as required. Students reach the most depth (the procept layer) when they can demonstrate flexibility in viewing a facet of a function as either a process or an object, as required by the problem situation.

To create student profiles, the two dimensions are combined diagrammatically with the layers as concentric circles representing increasing depth, sliced into sectors representing various facets. It is important to note that this visualization oversimplifies the complexity of the cognitive structure. Each facet may not be amenable to the various layers. Some facets, such as numeric, might be essentially more primitive than other facets, such as symbolic. Note also that the boundaries are porous. Facets flow into each other, such as the symbolic and no-



tational facets. Layers flow into each other such as the procedure and process layers. It may be at these porous boundaries where the most interesting information resides. The facets should be viewed as slices that can be moved around. An interesting area of research is the nature of the boundaries between various pairs of facets. Finally, we might add a third construct to layers and facets called **levels**. Each level might be a collection of layers and facets. Achieving the procept layer for one facet may be a precursor to moving into the procedure layer at the next level. For example, students often encounter evaluation of expressions prior to the function concept. Students may have already encapsulated the algebraic process of evaluation as a concept of expression. By applying this knowledge to function, they may move from the level of expression to the level of function.

Study

Students at four community colleges completed written function surveys at the beginning and at the end of a “reform” beginning algebra course during the spring semester of 1996. Subsequently, several students at each site participated in task-based interviews. One hundred forty-nine beginning algebra students completed a pre-course function survey. The post-course survey was completed by 82 students. The number of surveys in which students completed both pre- and post-course surveys was 70. Both the written surveys and interviews were used to profile students’ concept image of function. This paper mentions some preliminary results of the pre- and post-course surveys while analyzing the concept image of function held by one student, DB, using the framework described previously.

Selected Quantitative Results

In a series of 4 questions, students were given functions in the form of a function machine, an equation in two variables, a two-column table, and a graph. In each case they were asked to find a specific output, given the input and vice versa. They were given input and asked to find output in part a, while, for part b,

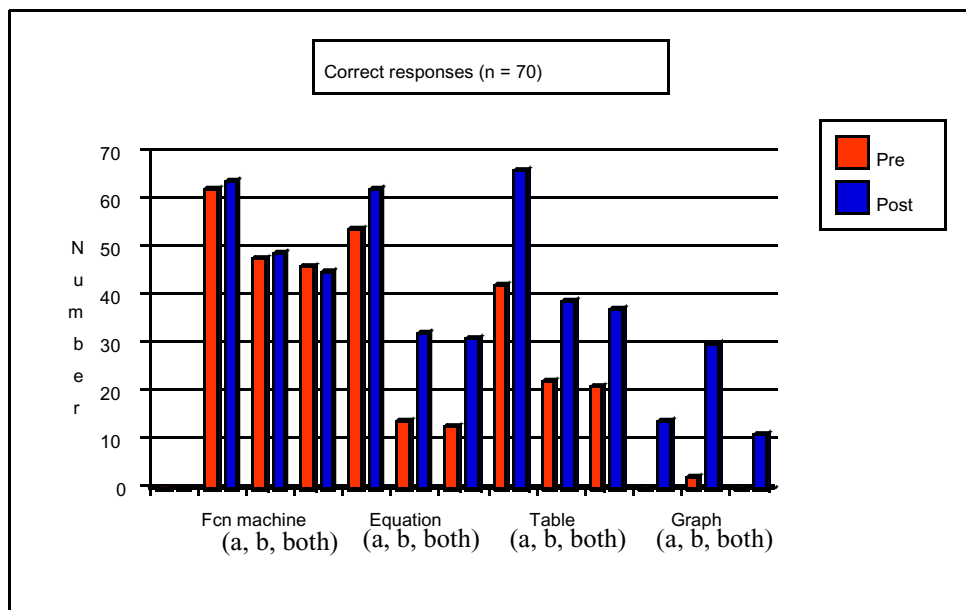


Figure 1

they were given output and asked to find input. Those that were correct on both parts were also counted. The bar graph displays the number of correct answers (Figure 1).

Both a Sign Test and a Wilcoxon Test were performed comparing the pre- and post-course performances of each student. Each question was scored using 1 point for a correct answer on part a and 2 points for a correct answer on part b. Thus, each question had a possible total of 3 points. The changes from pre- to post-course on the symbolic, numeric, and geometric facets were all significant at a 0.001 level. The data show little change using the function machine, but this may be somewhat due to the fact that a high percentage of students were able to correctly use the function machine on the pre-course survey. Not surprisingly, the geometric facet presented the most difficulty.

To measure the written facet, students were asked to define “function” on both pre- and post-course surveys. The bar graph in Figure 2 displays the results. The definitions provided were categorized based on the main focus of what the student wrote.

On the pre-course survey, 93% were unable to provide a satisfactory definition of function.

50% gave “colloquial” responses—that is, responses based on non-mathematical uses of the word. On the post-course survey, 52% wrote satisfactory definitions while only 23% still wrote answers categorized as “colloquial.”

Summarizing, the pre- and post-course surveys supplied information on development of student concept image of function in the colloquial, symbolic, numeric, geometric, and written facets. Students demonstrated significant growth from pre- to post-course in all facets except the colloquial (function machine). However, students demonstrated a high degree of comfort with this facet on the pre-course survey.

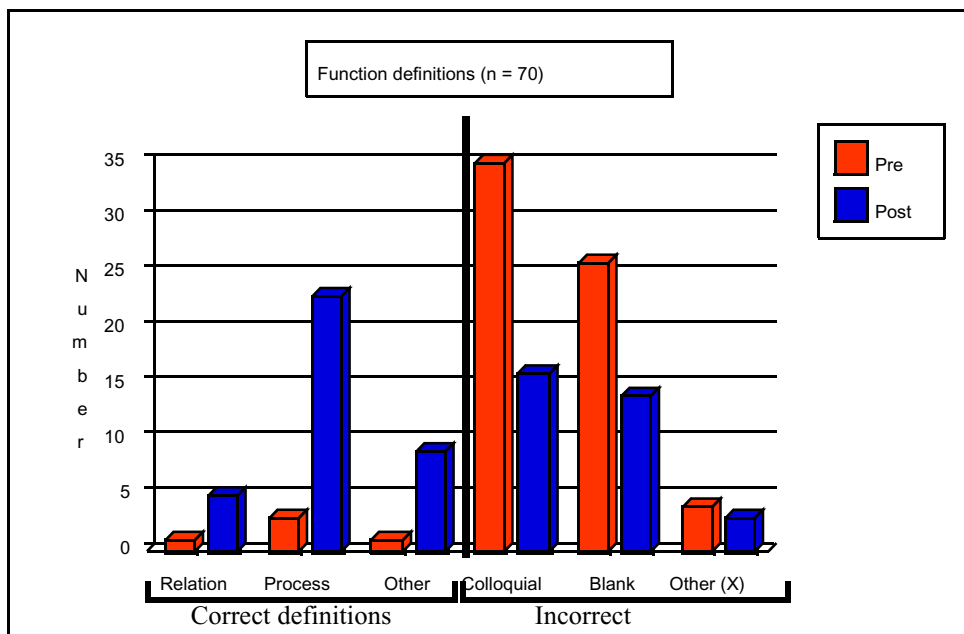


Figure 2

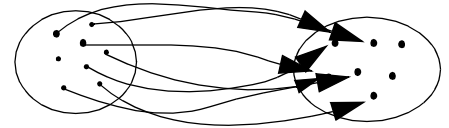
Selected Qualitative Results

To provide a flavor of the qualitative analysis, an analysis of a portion of the interview of one student, DB, is discussed. DB completed both surveys and participated in a two-hour task-based interview two weeks after the course concluded. DB is a female student in her mid-twenties. She had taken two years of high school algebra, barely scraping by. She was among the top students in the beginning algebra course discussed in this paper. Rather than focusing on all the facets, this analysis will focus on some of the

boundaries between facets.

The first boundary occurs between the written and verbal facets. How stable is a student's written definition of function as compared to her or his verbal definition? How easily does a student establish a conceptual connection between her or his definition and a new, alternate definition? DB was asked to write the definition of a function on both the pre- and post-course surveys. On the pre-course survey, she wrote: "A function is similar to an activity. A function in math would be to add or subtract some numbers" indicating she was at the pre-procedural layer at the beginning of the course. This definition would be classified as "colloquial." On the post-course survey, DB wrote: "A function is a relationship between two changing quantities." While she did not include the provision of a unique output for given input, she has moved to a more satisfactory definition. DB was asked during the interview to state her verbal definition of function and to compare this definition to other definitions. Her verbal definition matched the one she wrote on the post-course survey, demonstrating remarkable consistency. The following transcript probes her comfort level with alternate definitions.

Intvw: Consider the definition: A **function** is a **correspondence** that assigns to each element of one set one and only one element of a second set. A diagram appears at right. Discuss the relationship between **your** definition of function and this definition.



DB: Well, it's still putting one thing in and coming out with something different after going through some change or process.

Notice that DB states that "something different" comes out. How would she respond to the identity function? The interview continues.

Intvw: Okay so where is the input?

DB: This is the input (pointing to left circle).

Intvw: And the output?

DB: This is the output (pointing to right circle).

Intvw: Where would the process lie?

DB: Right here (pointing to arrows).

Intvw: Okay. Is that pretty much equivalent to your definition of function?

DB: Yes, I would say so.

Intvw: Consider the following definition: A **function** is a set of ordered pairs (a, b) in which for each value of a in the domain of the function, there is one and only one value of b in the range of the function. Discuss the relationship between your definition of function and this definition.

DB: I would say that they are the same definition. Given one input, only one possible output.

Notice that DB mentions uniqueness of output for a given input for the first time.

Intvw: What is acting as the input.

DB: Well the domain-*a*.

Intvw: And the output.

DB: *b*.

DB demonstrates good flexibility in her concept image of a function's definition. She appears able to identify the similarities between the definitions and easily merge them with her own definition. She is minimally at the process layer here and may well have sufficient understanding to be classified at the concept or even procept layer with respect to her verbal definition of function.


Other interesting boundaries occur between the symbolic, numeric, and geometric facets. This is one area that has received much curricular attention as a result of the widespread availability of graphics calculators (See the Rule of Three in Hughes-Hallett et al. 1992, for example). One last interview question explores the relationship between the symbolic, geometric, and numeric facets. The transcript is presented in full.

Intvw: An equation, a table, and a graph are displayed below for the same function. What is the output if

$y(x) = x^2 - 3x - 10$

X	Y1
-5	30
-4	18
-3	6
-2	0
-1	-6
0	-10
1	-12

WINDOW FORMAT
 Xmin=-10
 Xmax=10
 Xscl=1
 Ymin=-12
 Ymax=6
 Yscl=2



the input is -1 ? Did you use the equation, the table, or the graph to answer the question?

DB: -6 using the table.

Intvw: What is the output if the input is 4 ? Did you use the equation, the table, or the graph to answer the question?

DB: Now I would use the graph. 1, 2, 3, 4 (counting on the horizontal axis of the graph). Around -8 , I guess.

DB is very careful to look at the scale while answering this question.

Intvw: What is the output if the input is 12 ? Did you use the equation, the table, or the graph to answer the question?

DB: Um, I guess you'd have to just multiply that out since the 12 doesn't appear in either the table or the graph.

Intvw: What do you mean by "multiply it out?"

DB: Go up to the equation and input 12 for the x 's and figure it out.

Intvw: What is the output if the input is h ? Did you use the equation, the table, or the graph to answer the question?

DB: Um, well, again I mean all you would do is change the x 's to h 's.

Intvw: Would that be an acceptable output to you?

DB: Yes.

Intvw: What are the input(s) if the output is 0? Did you use the equation, the table, or the graph to answer the question?

DB: -2 using the table.

Intvw: Are there any others?

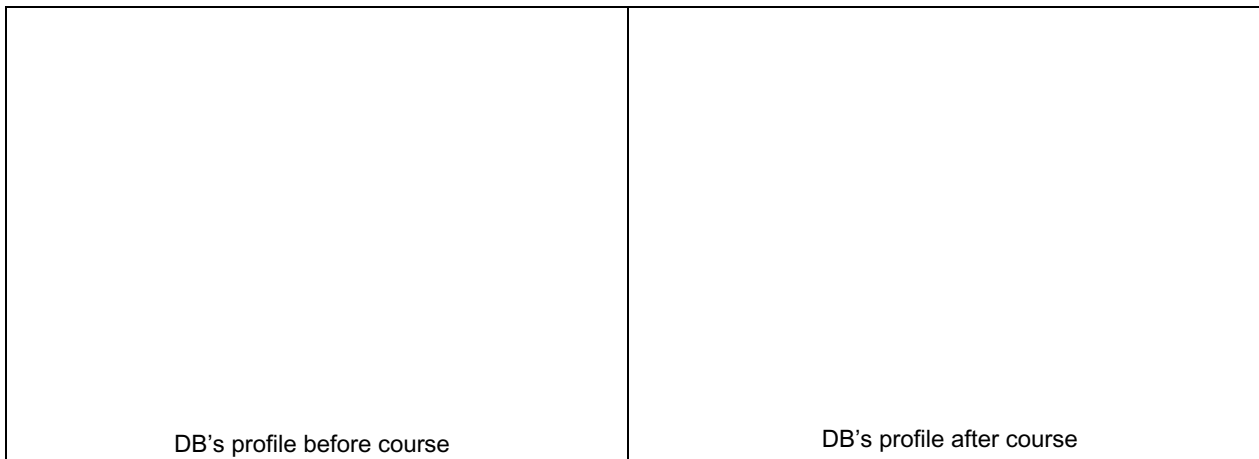
DB: Oh, obviously there would have to be (looking at the graph). Also, 5 on the graph.

Intvw: What are the input(s) if the output is 44? Did you use the equation, the table, or the graph to answer the question?

DB: There would be a possible negative or positive. You'd have to figure that out from the equation. You would replace y with 44 and solve for x .

It is impressive, watching the video, how easily DB moves between the various facets. She demonstrates equal facility in reading information off of the appropriate facet. In each case, she makes the most efficient choice. DB appears to have good mental connections between tables, equations, and graphs of functions. She is equally facile in moving between input and output in either direction.

Based on DB's written and verbal responses to the surveys and interview, a visual profile of her concept image of function is developed. While all of DB's responses were not discussed in this paper, the sample profiles provide a flavor of how the framework is used to visualize growth in concept image. The shading indicates the number of layers the student has demonstrated in her understanding of each facet. The student's knowledge of a specific facet has not been assessed if the outermost layer (pre-procedure) is unshaded. A profile of DB at the beginning of the beginning algebra course and at the completion of the interview is displayed. In essence, DB appears to be at least at the process layer for all facets, except pos-



sibly the kinesthetic, where more data must be collected. She has shown substantial growth as a result of the course. It is possible that she has reached the concept or even precept layer for several facets, but, again, more data is necessary to conclude this.

Reflection

The student population, including DB, for this research is a high-risk group who have had little prior success with mathematics. Using “function” as a focal point of their beginning algebra course, the authors hope to provide students with a vehicle to build meaning into their work with algebra. While common misunderstandings about function were apparent in the data, the results discussed in this paper suggest that function is not beyond the conceptual grasp of students at this level. The continued analysis of this data will be used to influence future curricular design and also to modify the theoretical framework to better model reality. How much depth can we expect for each facet is a follow-up question. Some of the more primitive facets, such as colloquial, numeric, and kinesthetic may not yield to as many layers as the symbolic or geometric facets. Which boundaries between which facets are the most porous. Which boundaries between which facets are the most difficult to penetrate? A follow-up research study is focusing on these questions.

References

- American Mathematical Association of Two-Year Colleges (AMATYC) (1995). *Crossroads in Mathematics: Standards for Introductory College Mathematics Before Calculus*. Memphis, TN: AMATYC.
- Ausubel, D. P. (1968). *Educational psychology: A cognitive view*. New York: Holt, Rinehart & Winston.
- Cuoco, Albert A. (1994). Multiple representations of functions. In Kaput, J. & Dubinsky, E. (Eds.), *Research Issues in Undergraduate Mathematics Learning*. (pp. 121–140). Washington, D.C.: Mathematical Association of America.
- Davis, R. B., Maher, C. A., & Noddings, N. (Eds.) (1990). Constructivist views on the teaching and learning of mathematics. *Journal for Research in Mathematics Education* Monograph Number 4. Reston, VA: National Council of Teachers of Mathematics.
- DeMarois, P. & Tall, D.O. (1996). Facets and layers of the function concept. In Puig, L & Gutierrez, A. (Eds.), *Proceedings of the 20th Annual Conference for the Psychology of Mathematics Education* Vol. 2. (pp. 297–304). Valencia, Spain.
- DeMarois, P., McGowen, M., & Whitkanack, D. (1996). *Mathematical Investigations: Concepts and Processes for the College Student*. New York, NY: HarperCollins College Publishers.
- Dubinsky, Ed & Harel, Guershon (1992). The nature of the process conception of function. In Harel, G & Dubinsky, E., *The Concept of Function Aspects of Epistemology and Pedagogy*. (pp. 85–106). Washington, D.C.: Mathematical Association of America.
- Dubinsky, Ed (1991). Reflective Abstraction in advanced mathematical thinking. In Tall, D.O. (Ed.), *Advanced Mathematical Thinking*. (pp. 95–124). Dordrecht, The Netherlands: Kluwer Academic Publishers.
- Gray, E. M. & Tall, D. O. (1991). Duality, ambiguity and flexibility in successful mathematical thinking. In Furinghetti, F. (Ed.) *Proceedings of the XV International Conference for the Psychology of Mathematics Education*. Vol. 1 (204–211). Tsukuba, Japan.
- Gray, E. M. & Tall, D. O. (1994). Duality, ambiguity, and flexibility: A “proceptual” view of simple arithmetic. *Journal for Research in Mathematics Education*, 25 (2). 116–140.
- Guralnik, D. B. Editor in Chief (1980). *Webster's New World Dictionary of the American Language* (Second College Edition). New York, NY: Simon and Schuster.
- Hiebert, James, & Carpenter, Thomas P. (1992). Learning and teaching with understanding. In Grouws, Douglas A. (Ed) *Handbook of Research on Mathematics Teaching and Learning*. New York: Macmillan Publishing Company. 65–97.
- Hughes-Hallett, D., Gleason, A. M. et al. (1992). *Calculus* Preliminary Edition. New York: John Wiley & Sons.

- Schwingendorf, Keith, Hawks, Julie, & Beineke, Jennifer (1992). Horizontal and vertical growth of the student's conception of function. In Harel, G & Dubinsky, E., *The Concept of Function Aspects of Epistemology and Pedagogy*. (pp. 133–149). Washington, D.C.: Mathematical Association of America.
- Sierpiska, A. (1992). On understanding the notion of function. In Harel, G & Dubinsky, E., *The Concept of Function Aspects of Epistemology and Pedagogy*. (pp. 25–58). Washington, D.C.: Mathematical Association of America.
- Simoneaux, Dolores & Kirshner, David (1994). Negative consequences of rote instruction for meaningful learning. In Kirshner, David (Ed.) *Proceedings of the Sixteenth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Volume 1). 218–224.
- Tall, D. O. (1996). Information technology and mathematics education: enthusiasms, possibilities and realities. Plenary address at the International Congress on Mathematics Education. Seville, Spain.
- Tall, D. O. (1991). Mathematical processes and symbols in the mind. Mathematics Education Research Centre. University of Warwick. Coventry, England.
- Tall, D. & Bakar, M. (1992). Students' mental prototypes of functions and graphs. *International Journal of Math, Education, Science, and Technology*, 23 (1). 39–50.
- Tall, D. O. & Vinner, S. (1981). Concept image and concept definition in mathematics, with particular reference to limits and continuity. *Educational Studies in Mathematics*, 12. 151–169.
- Vinner, S. (1983). Concept definition, concept image and the notion of function. *International Journal of Mathematical Education in Science and Technology*, 14. 239–305.
- Watkins, Anne *et al.* (1993). A survey of two-year college mathematics programs: the boom continues. *The AMATYC Review*, 14(2). 57.
- Yerushalmy, Michal & Schwartz, Judah L. (1993). Seizing the opportunity to make algebra mathematically and pedagogically interesting. In Romberg, Thomas A., Fennema, Elizabeth, & Carpenter, Thomas P. (Eds.), *Integrating Research on the Graphical Representation of Functions*. (pp. 41–68). Hillsdale, NJ: Lawrence Erlbaum Associates.