Functioning in the Real World: A PreCalculus Experience[1]

The Math Modeling/Precalculus Reform Project

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Abstract

The Math Modeling/PreCalculus Reform Project has developed, tested, and widely implemented an alternative to traditional precalculus/college algebra and trig courses that is in the spirit of the calculus reform movement. It is based on the applications of mathematics from the point of view of mathematical modeling. The course features innovative emphasis on mathematical topics such as families of functions, data analysis and fitting functions to data, difference equations models, modeling periodic phenomena, matrix algebra and its applications, and probability models. All are introduced to study and analyze a wide variety of applications from many different fields. In the process, students are encouraged to view mathematics and its applications from graphical, numerical, and symbolic perspectives with a focus on conceptual understanding. Student project assignments, writing, use of technology, and collaborative learning are all emphasized throughout. All of these are major themes that permeate both the NCTM Standards and the AMATYC Standards for Introductory College Mathematics Courses. The project text, *Functioning in the Real World: A PreCalculus Experience*, is published by Addison-Wesley and has been class-tested by approximately 200 instructors involving approximately 5000 students at universities, liberal arts colleges, engineering schools, two year colleges and high schools around the country.

Institutional Data

Suffolk Community College is a large comprehensive two year college located on Long Island, about 50 miles east of New York City. The three campuses together enroll over 21,000 students, the majority of whom are part time. The "typical" student is about 28 years old and female.

Most instructors require their students to have graphing calculators in all math courses from college algebra on up; some instructors require these calculators in their statistics and finite mathematics courses. The math department provides access to computers for students in its Math Learning Center. The college provides a number of open computer labs for students in central Computer Laboratories housed in each campus' library.

The other members of the project working group are Florence S. Gordon of New York Institute of Technology, B. A. Fusaro of Florida State University, Martha J. Siegel of Towson State University, Alan C. Tucker of the State University of New York at Stony Brook, and Judy Broadwin of Jericho High School in New York.

The project materials have also been class-tested at about 75 colleges, universities and high schools around the country. The technologies used at these institutions vary dramatically. Minimally, students in the different versions of the project course have used graphing calculators. At many of the class-test sites, the institutions provide dedicated computer labs with Derive, Maple, Mathematica, TEMath, spreadsheets, etc.

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[1] This paper is adapted from an article in *Exemplary Programs in Introductory College Mathematics* (edited by Susan Lenker, MAA Notes #47, Mathematical Association of America, 1998) which consists of articles on the winning projects in the INPUT (Innovative Projects Using Technology) competition sponsored by the Annenberg Foundation/Corporation for Public Broadcasting.

The Project

Motivation for the Project

The calculus reform movement has made great strides in changing the emphasis in calculus to emphasize graphical and numerical ideas in addition to symbolic manipulations. Some of the major themes in the reform movement include focusing on student projects, realistic applications, the use of technology, and collaborative learning. In the MAA's ACRE (*Assessing Calculus Reform Efforts*) report, Tucker and Leitzel [1] indicate that about two-thirds of all colleges and universities have implemented moderate or large scale reform efforts. The results will hopefully transform calculus into a *pump, not a filter*.

The next step is to address the problem of how we "fill the tank": how we increase the numbers of students who proceed on into the calculus portion of the mathematics pipeline. Each year approximately 700,000 college students take some variety of precalculus course; yet only about 15% to 20% of them ever go on to *start* calculus. Most of those who do go on to calculus display a singular lack of retention of the material they were taught and often cannot successfully complete calculus. The standard precalculus courses neither motivate the students to go on in mathematics nor adequately prepare them if they do continue. These issues are the subject of the MAA Notes volume *Preparing for a New Calculus* [2].

However, as the reform calculus courses are more widely adopted and implemented, it becomes evident that we need to change the preparatory courses that lead toward calculus. The new calculus courses place very different expectations on the students who are required to do very different things in very different ways. Consequently, the preparation for calculus should emphasize the same themes.

The traditional courses at the precalculus and college algebra level are designed primarily to develop algebraic skills that once were essential for success in later courses. The wide availability of technology and the changing requirements, especially in the client disciplines, requires a rethinking of this paradigm. For the results of a series of interviews with leading educators in the client areas, see Gordon [3]. Currently, students in upper division courses in engineering and the sciences do relatively little with pencil and paper mathematics; instead, they focus on developing mathematical models to describe real-world phenomena. These models typically involve differential equations, matrix methods, or often probabilistic simulations. The students examine the behavior of the solutions, particularly as the parameters underlying the phenomena change.

Simultaneously, students in business, the social sciences, and the biological sciences are expected to recognize trends from sets of data, construct appropriate mathematical models to fit the data, and make corresponding predictions based on the models developed. This is actually remarkably similar to what students in lab courses have been doing for centuries; the difference is that the students in the business and social science courses typically use spreadsheets for the analysis rather than hand-drawn graphs.

Therefore the primary emphasis on algebraic manipulation in traditional preparatory mathematics classes does not provide the foundation that students now need for all of these disciplines, nor does it adequately prepare them for the new calculus. Instead, a broader preparation is needed, one that better reflects the practice of mathematics. Students must learn to:

1. Identify the mathematical components of a situation (i.e., model it).
2. Select the right tool (paper-and-pencil, graphing calculator, CAS package, spreadsheet, etc) to solve the problem.
3. Interpret the solution in terms of the original situation and, if necessary, change the assumptions used (i.e., introduce additional factors) in the model.
4. Communicate the solution to an individual who likely knows less mathematics, but who pays the salary.

From this point of view, it is clear that

> *No college graduate will be paid $30,000 per year to solve problems whose solutions were memorized in high school or college mathematics courses.*

For that matter, given the existing technology (such as the TI-92 calculator),

> *No college graduate will be paid $30,000 per year to be nothing more than a poor imitation of a $150 calculator*! *Yet, no matter how much we drill our students, we will never make them as fast or as accurate as that calculator.*

It is essential that we aim to produce something far more valuable than an incomplete organic clone of a calculator or computer program.

Project Data

The Math Modeling/PreCalculus Reform Project, with major funding from the NSF's Division of Undergraduate Education, addressed this challenge by developing, testing, implementing and disseminating information about a dramatically different alternative to standard precalculus and college algebra/trigonometry courses. The project was under the direction of Sheldon P. Gordon and B. A. Fusaro; Gordon is the principal author of the project materials, *Functioning in the Real World: A PreCalculus Experience* [4]. As mentioned above, the members of the Project's working team included Florence S. Gordon, Martha J. Siegel, Alan C. Tucker, and Judy Broadwin.

The *Functioning in the Real World* course extends the common themes in most of the calculus reform projects, as well as the principles expressed in both the NCTM Standards and the AMATYC Standards. The course focuses on mathematical concepts, provides students with an appreciation of the importance of mathematics in a scientifically oriented society, provides students with the skills and knowledge they will need for subsequent mathematics and other quantitative courses, makes appropriate use of technology, and fosters the development of communication skills. Our goal is to emphasize the qualitative, geometric and computational aspects of mathematics within a framework of mathematical modeling at a level appropriate to precalculus students.

We capitalize on the fact that most students are more interested in the applications of mathematics than in the mathematics itself, so that the applications drive the mathematical development. All the mathematical knowledge and skills that students will need for calculus are introduced and reinforced in the process of applying mathematics to model realistic situations and solve interesting problems that arise naturally from the contexts. We have found that this approach excites the students and encourages them to go further with mathematics by showing them some of the payoff that mathematics provides.

Our materials and the course based on them serve a multiplicity of audiences:

- A one semester course that lays a different, but very effective, foundation for calculus;
- A one or two semester course that stands as a contemporary capstone to the mathematics educa tion of students who do not plan to continue on to calculus. We have seen that the course encourages many of these students to change their minds and to go on to calculus and other quantitatively related courses.
- A course that provides the foundation for further courses in discrete mathematics and related offerings.

Innovations

Our support from the NSF gave us the opportunity to create an entirely new course at the precalculus or college algebra/trigonometry level from scratch. We were not constrained to produce a slight variation of existing texts or courses, but rather to take a fresh look at:

- the mathematical topics that are being used in the reform calculus courses,
- the capabilities of current technology,
- the type of modern mathematics that is being used in other disciplines and on the job,
- interesting and realistic applications that can be used to motivate and drive the mathematical development,

- the different learning environments that have been emerging both in reform calculus and in the high schools.

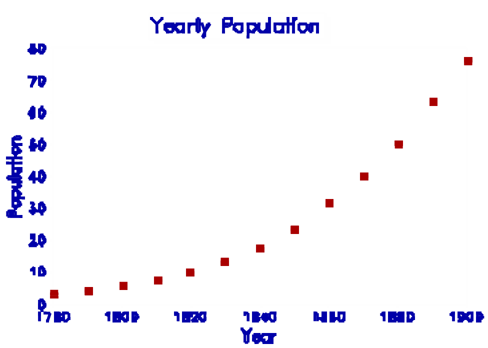
The challenge we faced was to incorporate all of these principles into a coherent course that would be meaningful, motivating, and accessible to students who traditionally are poorly prepared for college level mathematics. We outline several of our responses to this challenge as follows.

• **Fitting Functions to Data** Almost anyone using mathematics today begins with a set of data, either a set of experimental measurements from the laboratory or information on some quantity of interest. The problem faced usually is to identify any trend or pattern, find an appropriate mathematical model to describe the process, and use it, if it is a good fit, to make predictions (interpolate or extrapolate) about the process. The simplest case is fitting a line to a set of data in the least squares sense. We focus on this because it is so fundamental an idea, one that is used throughout the physical, biological and social sciences. It also provides the opportunity to study linear functions in detail in realistic settings.

However, we also look at the question of fitting a non-linear function to a set of data that is clearly not linear. For instance, consider the following data on the growth of the U.S. population (in millions) from 1780 through 1900:

Year	Population	log(Pop'n)
1780	2.8	0.447
1790	3.9	0.591
1800	5.3	0.724
1810	7.2	0.857
1820	9.6	0.982
1830	12.9	1.111
1840	17.1	1.233
1850	23.2	1.365
1860	31.4	1.497
1870	39.8	1.600
1880	50.2	1.701
1890	62.9	1.799
1900	76.0	1.881

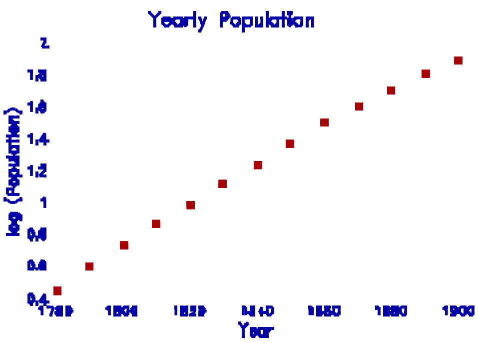
If you examine the ratios of successive population terms, you will observe that they are roughly equal, which indicates that the population values grow approximately exponentially. Figure 1 is a plot of the actual data values which shows the apparently exponential pattern.



When you suspect that a certain phenomenon follows an exponential pattern of the form $P(t) = Bß\, a^t$, then

$$\log P = \log B + t \log a.$$

That is, log P is a linear function of t and so we should expect that a plot of log P versus t should be linear. Figure 2 shows the associated plot of the transformed data with log (P) as a function of t. It is clear that the transformation has linearized the data.



Using the ideas of linear regression analysis previously developed, students now find the line that best fits this transformed data; their calculator or computer program tells them it is generically of the form:

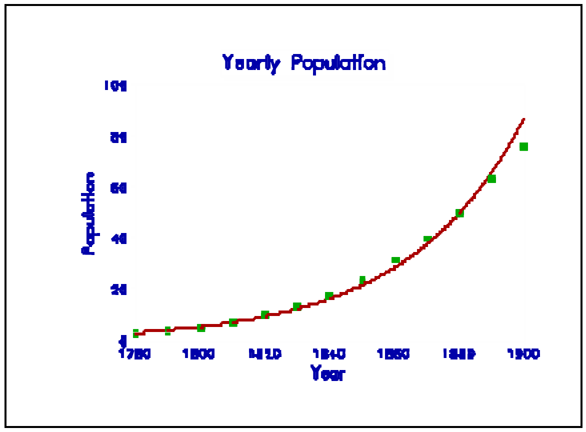
$$Y = .121X + .487,$$

but they must interpret this in terms of the actual variables used as

$$\log P = .121t + .487 .$$

They then must undo the original transformation by applying the inverse function and all the pertinent operations to obtain

$$\begin{aligned} P &= 10^{\log P} \\ &= 10^{0.121t + 0.487} \\ &= 10^{0.121t}\text{ß}10^{0.487} \\ &= (10^{0.121})^t\text{ß}(10^{0.487}) \\ &= 3.069\text{ß}(1.321)^t \end{aligned}$$



Notice the level of algebra needed to undo this transformation. Interestingly, the students do not complain because they are doing the work in the context of answering questions of interest, not merely to develop some algebraic skills that they see no need to possess. We show how well this exponential function fits the original population data in Figure 3.

Incidentally, notice that the base, or growth factor, for this exponential function is 1.321, so that the corresponding growth rate is 32.1% per decade, or somewhat over 3% per year, as we anticipated. Also, we analyzed the population values only up through 1900 because the rate of growth has diminished considerably during the 20th century (it is currently about 0.7% per year). We come back to study the U.S. population over the entire 1780-1990 period in the context of discussing logistic, or inhibited, growth.

In a comparable way, if one suspects that a set of data values follows a power function of the form $Q = B\text{ß}\, x^p$, then $\log Q = \log B + p \log x$, which means that $\log Q$ is a linear function of $\log x$. Therefore the data can be linearized by plotting $\log Q$ versus $\log x$ and the linear regression analysis technique can be used to find the equation of the best fit linear function. The students then need to undo the transformation using all of the usual properties of exponential and logarithmic functions. Again, the level of manipulation is far greater than one would normally expect with simple artificial problems, but the students are willing and able to rise to the occasion.

• **Modeling Periodic Behavior** Periodic phenomena abound in the real world, but they cannot be modeled mathematically by functions as simple as $y = 3 \sin 2x$. For example, the number of hours of daylight on a given day of the year at any particular location is a periodic function of time. If the location is San Diego, say, then the number of hours of daylight can be modeled by

$$H(t) = 12 + 2.4 \sin\big((2"/365)(t - 80)\big),$$

where t is the number of days from January 1 of any given year. What do the different parameters mean? The 365 clearly represents the number of days in a year, or the length of a cycle. The 12 represents the average number of hours of daylight, which occurs on the spring and fall equinox. The 2.4 represents the maximum variation above and below the middle value, so the longest day in San Diego has 14.4 hours of daylight and the shortest day has 9.6 hours. What about the 80? Since it is related to the variable t, it must represent some particular date and, if you count off days, you will find that the 80th day of the year is March 21, the spring equinox, when exactly 12 hours of daylight occur.

In the process of analyzing these parameters, the students achieve a much deeper understanding of what amplitude, period, frequency, vertical shift, and phase shift signify because the terms arise in a meaningful context.

Once such a function is available, it is possible to ask a variety of pertinent questions, such as: How many hours of daylight would you expect on a particular date? When will there be 13 hours of daylight in San Diego? Or, we can assign the students a project to determine the comparable formula for the number of hours of daylight in any other city -- all they need is to find the data on sunrise and sunset on one day of the year, preferably the longest or shortest.

• **Modeling with Difference Equations** Difference equations can be used to model discretely almost any process that can be studied continuously using differential equations, but at a level requiring no more than standard algebra topics. For example, suppose that 500 pounds of a contaminant are initially dumped into a lake and 10% of it is washed away each year. We denote the level of contaminant in the lake after n years by C_n . If 10% is washed away during that year, then 90% remains at the end of the year, so that the level the following year is modeled by the difference equation

$$C_{n+1} = (0.9)C_n,$$

starting with the initial contaminant level $C_0 = 500$. The corresponding solution is

$$C_n = 500\,(0.9)^n,$$

for all values of n Æ 0. This expression can be found either algebraically in an iterative manner or by more formal solution methods; the graph of the solution can be found directly from the difference equation using the difference equation capabilities of many graphing calculators or by graphing the solution function. This solution is an exponentially decaying function that tells us that the level of contaminant will slowly decay down to zero over time.

Next, suppose we change the underlying scenario in a variety of ways:

1. In addition, a factory annually dumps 100 pounds of the contaminant into the lake.
2. The plant increases its production yearly and so increases the amount of the contaminant it dumps by 25 pounds each year starting with the initial level of 100 pounds from scenario (1).
3. The plant increases the amount of the contaminant dumped into the river by 20% per year starting with the initial level of 100 pounds from scenario (1).
4. EPA regulations require that the plant reduce the level of dumping by 25% a year.

The difference equations corresponding to each of these situations are as follows:

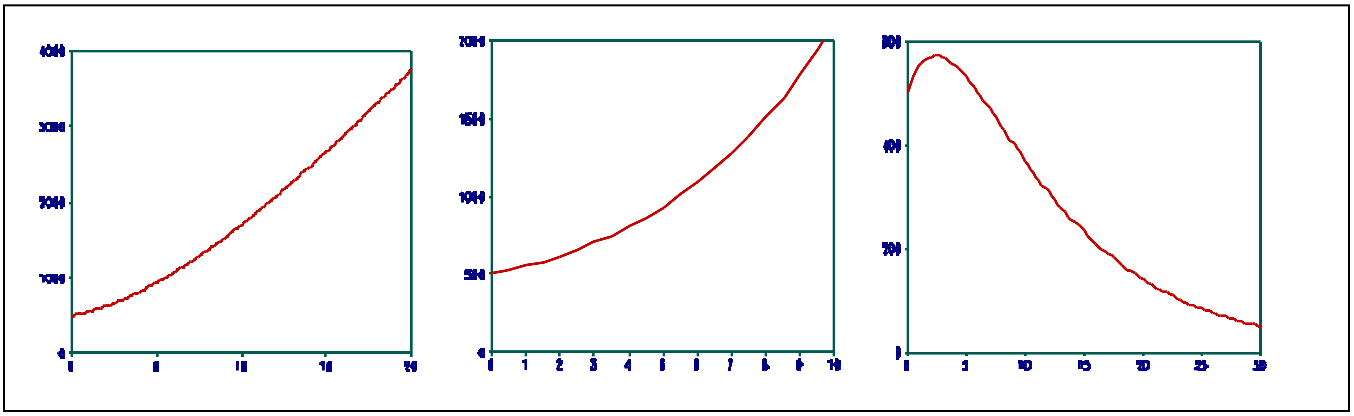
1. $C_{n+1} = 0.9C_n + 100, \quad C_0 = 500$
2. $C_{n+1} = 0.9C_n + 100 + 25n, \quad C_0 = 500$
3. $C_{n+1} = 0.9C_n + 100\,(1.20)^n, \quad C_0 = 500$
4. $C_{n+1} = 0.9C_n + 100\,(0.75)^n, \quad C_0 = 500$

Notice how easy it is to introduce the different assumptions. The corresponding closed form solutions, which are all found using the discrete analog of the method of undetermined coefficients (a marvelous way to surreptitiously drill some important algebraic skills, the identical ones they will need for solving differential equations), are:

1. $C_n = 1000 - 500(0.9)^n$
2. $C_n = 2000(0.9)^n - 1500 + 250n.$

3. C_n = 166.7 $(0.9)^n$ + 333.3 $(1.2)^n$.
4. C_n = 1166.7 $(0.9)^n$ - 666.7 $(0.75)^n$.

How do the various solutions behave? Solution (1) is fairly obvious; the decaying exponential term dies out and so the level of contaminant slowly rises toward a horizontal asymptote of 1000. However, the others require a bit of analysis. We show their respective graphs in Figures 4-6. In the process of describing the observed behavior and relating it to the terms in the solutions, students develop a deeper understanding of the behavioral characteristics of each of the component functions.



It is also possible to raise related questions. For instance, in Figure 6, can you estimate the maximum level achieved by the contaminant? What is the effect of changing the initial value C_0? What happens to the solution if values for the other characteristics change, such as the percentage of the contaminant washed out each year or the rate at which the company is required to reduce its contamination?

Status of the Project

Preliminary and draft versions of the project materials have been class-tested by about 200 faculty at almost 75 schools around the country These institutions involved cover the full spectrum, including large research universities, four year liberal arts colleges, engineering schools, two year colleges, and high schools. Each subsequent revision of the materials has reflected the suggestions and experiences of the class-testers. The official first edition containing a portion of the project materials was published in November, 1996. Both the preliminary editions and the first edition are accompanied by computer software developed under the project to provide both computational tools and the ability to experiment with most of the mathematical models studied.

Project Report

Content

The fundamental idea around which the *Functioning* course is based is the function concept in all of its manifestations with emphasis on its applications to the world around us. The mathematical models developed in the course are based primarily on data analysis and curve fitting, difference equation models, models for periodic phenomena, probability models, and matrix algebra.

Curve fitting techniques are introduced early and used throughout so that students learn how to interpret data values that arise in many different contexts. The data analysis methods simultaneously provide immediate reinforcement regarding the behavior and properties of the different families of functions studied. The focus on difference equations includes applications of first and second order difference equations and systems of first order difference equations. The emphasis is on modeling a variety of situations, interpreting the behavior of the solutions in terms of the situations, and looking at the effects of changes in parameters. Probabilistic ideas are interwoven throughout the course in the context of performing random simulations and studying geometric probability. Matrix algebra is introduced as a unifying tool for investigating a wide array of applications including systems of difference equations, Markov processes, regression analysis, and geometry.

As indicated above, the trigonometric functions are introduced from the point of view of modeling periodic phenomena. For example, students are asked to construct a sinusoidal function to model the temperature in a house where the thermostat is set to turn on the furnace when the temperature drops to 66% and to turn it off when the temperature reaches 70%, and this cycle repeats every 20 minutes. The students are then asked how they would change the model if the house were located in different climate areas. As another example, they are asked to model a person's blood pressure over time given readings of 120 over 80 and a pulse rate of 70.

The *Functioning* course involves a considerable degree of computer or graphing calculator work to explore the implementation of most of the mathematical models. It involves a variety of live classroom experiments to investigate the accuracy of the mathematics in predicting the results of actual processes or to help develop mathematical models based on observed experimental data. The course also features a series of student investigations to provide a real-life dimension to the mathematics.

The following is an annotated description of the contents of the project materials.

Functions in the Real World introduces students to the function concept from graphical, numerical, symbolic, and verbal points of view (which are used throughout the course) as functions arise in daily life. The emphasis is on the behavior of functions (increasing or decreasing, concave up or concave down, point of inflection, periodicity, etc). Probabilistic ideas are also introduced via Monte Carlo methods.

Families of Functions includes linear, exponential, power, and log functions, and inverse functions, with emphasis on their applications and their behavior (growth, decay, relative growth rates, concavity, etc), very much in the spirit of the calculus reform movement.

Fitting Functions to Data includes linear and nonlinear curve fitting to reinforce the properties of the different families of functions, to develop algebraic skills in working with the properties of those functions, and to connect the mathematics to the real world.

Extended Families of Functions includes polynomial functions, fitting polynomials to data, the nature and relative frequency of the roots of polynomial equations, building new functions from old (shifting, stretching, sums, differences, products, quotients, and composition of functions) , finding roots of equations, and finding polynomial patterns (including sums of integers and sums of squares of integers).

Sequences and Difference Equations includes the development and analysis of models for describing population growth, logistic (inhibited) growth, eliminating drugs from the body, radioactive decay, Newton's laws of heating and cooling, geometric sequences and their sums.

Modeling with Difference Equations includes first order non-homogeneous difference equations, fitting quadratic functions to data, Newton's laws of motion, modeling the stock market with a stochastic model, financing and amortization, fitting logistic curves to data, iteration and chaos, etc.

Modeling Periodic Behavior includes using trigonometric functions to model phenomena such as the number of hours of daylight as a function of day of the year, temperatures over the course of a year, and tides; relationships between trig functions; approximating periodic functions with sine and cosine terms; approximating the sine and cosine functions with polynomials; properties of complex numbers; and chaotic phenomena.

Second Order Difference Equations includes the Fibonacci sequence, transmission of information, mechanical systems with simple and damped harmonic motion, non-homogeneous models including a national income model and an inventory analysis model.

Matrix Algebra and its Applications includes a variety of applications of matrices, such as Markov chains in the spirit of a finite math course, not merely the use of matrices for solving systems of linear equations.

Probability Models includes binomial probability and the binomial expansion, geometric probability, estimating areas of plane regions using Monte Carlo simulations, waiting time models, the spread of epidemics, and random patterns in chaos.

Systems of Difference Equation includes the predator-prey model, an arms-race model, a labor-management model, a model for marriage rules among the Natchez Indians, and matrix growth models leading to a treatment of the eigenvalue/eigenvector problem.

Geometric Models includes analytic geometry, the conic sections, parametric curves, the average value of a function, and curves in polar coordinates.

Pedagogy

The *Functioning* course almost forces a change in pedagogy; it is virtually impossible to give the course in a formal lecture format. The non-routine nature of many of the problems make them ideal for having the students attack them in small groups using collaborative learning. Alternatively, the problems can be approached by the entire class (if not too large) as a group with the instructor being the group leader who asks leading questions and who is more of a recorder at the board than the dispenser of all mathematical knowledge.

We also emphasize the use of individual or small group projects related to the mathematical content of the course. For example, students might be required to find a set of data of interest to them and perform a complete analysis of it -- finding the best linear, exponential, and power function to fit the data, and asking and answering pertinent questions (i.e., predictions) based on the context. Each student would then be required to write a formal project report. For instance, during the current semester, a sample of the topics my students studied include:

The number of sexual harassment cases filed as a function of time.

The likelihood of car crashes as a function of blood alcohol level.

The growth of the prison population as a function of time.

The time of high tide at a beach as a function of the day of the month.

The amount of solid waste generated per person as a function of time.

The time for water to come to a boil as a function of the volume of water.

The size of the human cranium over time during the last three million years.

The results of a serial dilution experiment in biology lab.

The growth in the Dow-Jones average as a function of time.

The Gini Index measuring the spread of rich versus poor in the population over time.

The number of immigrants who entered the U.S. over time.

The mean annual income as a function of the level of education.

As another example of a project, students are given a set of data of a periodic nature, say historical high temperature readings in a given city every two weeks, and are asked to create a sinusoidal function that models the temperature. This can be quite a challenging problem at first glance, and yet in the process the students really come to understand the meaning of all the parameters in the general equation for a sinusoidal function. A particularly effective way to start this is to have the students begin in groups of three or four, so that someone has to speak up and make some initial suggestion of where to begin. Eventually, I have each student complete the analysis independently and prepare a formal written report.

The fascinating thing about such a project is that there are quite a number of different strategies that can be developed for estimating the various parameters and different students come up with them. One of the most memorable lines by a student that I have ever read appeared in one such report: "*The next quantity to be determined is the frequency. This was deceptively simple.*" How often does a student in a precalculus course describe the frequency of a sine function as *deceptively simple*, particularly when the value he obtains for the frequency is .0172 or 2"/365?

The use of such projects, however, raises several important questions. First, how do we grade such things? Obviously, there are no right or wrong answers. There should be certain things that we would expect to see, such as correct mathematical formulations and calculations, and it is relatively easy to devise a grading scheme to account for that. However, there is considerable variation from one student to the next in the quality of work they do, and this is much harder to assess. Some students turn in project reports that are truly outstanding, almost professional in nature, and I personally feel that such extra effort should be rewarded in some fashion. Also, special attention should be paid to students for whom English is not the first language.

The second question is, how do we factor this in as part of the overall assessment of a student? In my case, I typically assign three or four such projects during a semester, and count them together as the equivalent of two class tests. In all, I usually give three class tests plus a cumulative final that counts as the equivalent of two other tests, so the projects amount to roughly 2/7 of the final average. Considering the amount of mathematics that the students learn in the process of doing these projects, this seems quite reasonable.

We also encourage the use of "live" experiments in the classroom to collect data to be analyzed. One such activity, using a CBL unit and calculator to study the height of a bouncing ball, is described in the next section. Others might include using a thermometer to measure the temperature of water being brought to a boil as a function of time or the height of liquid in a container having a hole at the bottom (Torricelli's Law) as a function of time.

The Role of Technology

We believe that technology has tremendous implications for the teaching and learning of mathematics. The problem we as educators face is how to use the available technology in the service of the mathematics rather than as an end in itself. To illustrate this dichotomy, consider the question of fitting functions to data. Most graphing calculators have the ability to find the best exponential, power or log function to fit a set of data; they also fit the best quadratic, cubic or quartic polynomial; some can fit the best sinusoidal or logistic function to a set of data. All of this can be accomplished literally at the push of a single button.

However, we firmly believe that pushing that button, at least early on, is a mistake for most precalculus or college algebra students. Rather, we ask that students examine the original set of data, look for a general pattern, appropriately transform the data entries to linearize them, use a calculator or computer to obtain the best linear fit to the transformed data, and then undo the transformation using the algebraic properties of the appropriate inverse function. In this way, the students are learning more mathematics while developing some of the manipulative skills they will need to succeed in calculus and other courses.

The *Functioning* materials were developed from the point of view that each student has a graphing calculator. We do not focus on any particular make or model, however, preferring to keep the descriptions of calculator usage as generic as possible. In addition, the project has developed a package of over 40 computer graphics programs in BASIC for IBM and compatible computers that allow students to experiment with most of the models covered in the course. A multimedia package for both IBM and Macintosh machines is currently under development.

The experiences reported by the class-testers reflect a full spectrum of technology scenarios. Almost all have required a graphing calculator. Some have incorporated a computer laboratory component into the course in which each student has access to and uses a powerful mathematical package such as Derive, Maple, TEMath, or the software package developed under the project. Several have had their students use spreadsheets such as Excel. Several have used the CBL unit to collect and analyze data.

For instance, I typically start the first class by bringing in a basketball, dropping it, and asking the students to sketch the graph of the height of the bouncing ball as a function of time. I have the students compare their graphs with their neighbors in small groups to "break the ice" about talking mathematics to one another.

I then ask for several volunteers and have them run through the same experiment using the CBL unit connected to a calculator with a viewscreen to collect and display the actual data on height as a function of time. I trace the curve on the calculator, indicating the points where interesting behavior occurs -- when the ball bounces, when the ball reaches its maximum height, and what those maxima are. I create a table on the blackboard summarizing those results and investigate them for patterns -- the times of the ball bounces and the times when it reaches its maximum height follow roughly linear patterns (though the terminology is certainly not available yet) and the values for the maximum heights decay in a decidedly nonlinear pattern. Finally, I isolate one arch of the graph and, using the statistical features of the calculator, fit the best parabola to the curve. All of these activities dramatize to the students that this is not your usual mathematics class. The activities also preview for the students many of the major mathematical themes that will come up during the course.

One of the inherent problems with technology, particularly graphing calculators, is dealing with the growing variety of options available. An instructor can ask for a particular calculator model, but what happens when students come into the course with a different model that they used in high school (a scenario that will become far more prevalent in the next couple of years) or with a used calculator that they bought cheaply from another student who posted a sign in the hallway? Similarly, an instructor may decide to use a particular piece of software with the course, but what of the students who have learned to use a different piece of software in some other course? If it is just a matter of one spreadsheet versus another, there is no real problem. If it is a spreadsheet instead of a mathematical package such as Derive, say, there may be greater problems, particularly when the other software may do a better job at some chores (such as creating fancy graphs based on data) than the designated one.

Another question to be considered is: how does an instructor teach the technology? Should it be in class, thereby taking a goodly amount of time away from the mathematics? Is it possible for the math department to arrange for a series of technology training sessions, open to all students taking one or a group of related courses, during common hours?

Still another question to be considered is how the technology choices made by an instructor or the entire math faculty compare with the decisions being made by other departments. We suggest that instructors discuss their needs and course requirements with faculty from other fields to try to come up with reasonably uniform goals. Otherwise, one can easily find oneself in the position where the physics department has required an HP calculator, the chemistry department a TI calculator, and the math department a Sharp, and everyone, student and instructor, is frustrated and confused. Actually, we suspect that the choice made in the local high schools will soon trump any decisions made in the colleges.

Day to Day Mechanics

The innovative nature of the *Functioning* course places special burdens on both the instructor and the students. However, our experiences (see Section F below) indicate that this additional time and effort is well worth investing.

Because the course has so much content that is new to many instructors, it requires considerably more preparation time than a more traditional course where one can often walk in with virtually no preparation. Naturally, by the second and certainly by the third time one offers the course, it does become far less demanding. Similarly, the non-routine nature of the problems requires that considerably more attention be paid to selecting the ones to assign, as well as the number of problems to be assigned. It certainly does not make sense to say “Do problems 1 through 49, odd”. Also, we don’t recommend walking in “cold” and asking if there are any questions on the homework; many of them require some advance thought.

By their very nature, many of the problems admit of different interpretations. It is a delight to see students coming up with those interpretations. Also, most of the problems can be solved in a variety of ways -- analytically, graphically, numerically, and so forth. Consequently, the problems themselves almost automatically change the day-to-day classroom dynamics. Such problems can be used as the basis for cooperative or group learning, if an instructor so desires. Alternatively, the problems can be used in large-scale discussions lead by the instructor as springboards to introduce new ideas and to get the students to think mathematically. Probably the least satisfactory way to approach these problems is to walk in and present polished solutions on the board.

Another challenge faced by many instructors is a re-orientation. In traditional courses, the so-called applications tend to be quite artificial. We suspect that most students realize this, which accounts, in large measure, for their not seeing mathematics as a useful subject. In the *Functioning* course, however, one is almost forced to look for truly realistic examples and problems. Certainly, if one is using real data, the situation is inherently realistic. Sources of such examples tend to be non-traditional also, at least for mathematics. We urge both instructors and students, for instance, to look at the Statistical Abstracts of the United States, any information almanac, and even the Old Farmer's Almanac for data on a variety of periodic processes that can be modeled with sinusoidal functions.

The emphasis on realistic applications also tends to reduce the role of the instructor as the all-knowing expert in the room. There will always be several students in any class who are far more knowledgeable about electronics or automotive mechanics or chemical reactions or economic trends. The instructor must be willing to admit to being less informed and drawing such students out (though perhaps turning them off can be a bigger problem) to add to the real dimension of the mathematics. But that is what the course is all about -- convincing the students that mathematics applies to all areas of their lives. And when they make that connection, the course really comes alive for them!

Still another challenge is creating new, innovative problems for tests or quizzes. One advantage to having students do projects is that the instructor gains a wonderful pool of examples for subsequent use. (However, the first time through the course, before one has that pool, things are more difficult.)

From the students' perspective, this is also a very different course. We have found it is essential to explain to them, beginning on the first day, why they are being given such a different experience. We admit to many of the problems with more traditional courses and we use many of the same rationales we discussed above for the changing process of doing mathematics in practice. We discuss what technology can do for the students and what they must be able to do that technology cannot do: to think! Furthermore, we repeat many of these messages often throughout the course, so it is not just part of day one background.

Also, we recognize that the students are undergoing a major re-orientation in their view of what mathematics is and how one does mathematics. This is not something that is completed in a day or a week. Students likely buy into the new philosophy the first day or during the first week or so. However, this re-orientation in their thinking typically takes three or four weeks to sink in, so that they become reasonably comfortable with using a variety of mathematical tools (graphing, numerical approaches, algebraic approaches) to solve problems, with using some form of technology, with problems that do not have unique solutions that can be found in only one way, with the use of a variety of different letters (not just x and y) for variables, with thinking about what they are doing, and with discussing mathematics with others.

Often, students who have the strongest traditional math backgrounds (those who have the best algebraic skills) are the ones who are most resistant to this re-orientation. At the other extreme, students whose algebra skills are weak, who previously would be assessed as having little or no mathematical ability, often demonstrate incredible levels of mathematical insight using verbal, graphical, or numerical approaches.

Occasionally, there are some students who simply "do not get it", at least in the classroom. We have found that, with such individuals, it is necessary to drag them to the office for a one-on-one session to reinforce what some of the underlying ideas are. This one session often is all that is needed to bring these students around. Even if it isn't quite sufficient, they are far more likely to come back to the office for help on their own.

Finally, if one is offering a truly different course with different expectations of the students, it is critical to give exams that mirror the new philosophy. Students assess the apparent importance of things by what is tested. If the tests are mostly routine "solve the following" problems with a few template problems tossed in, the instructor will not truly change the culture. On the other hand, it is expecting a lot to give students several "new" problems and expect them to come up with novel solutions under the pressure of an exam. A set of class tests and the final exam for one semester are attached at the end of this article. Our experience is that most students taking this course are capable of far more than we have previously expected. Yet, it is asking a lot of all students. One way we have compensated for this is to grade tests out of about 110 points to give a little leeway. Another is to use projects to provide additional assessment alternatives.

Assessment

Most of the information we have received from the class-testers has been anecdotal in nature, but is often highly indicative of what goes on in their classes. For instance:

• Joe Fiedler and Ignacio Alarcon of California State College at Bakersfield report that of 45 students (a majority of whom were minority students) who started a two-quarter sequence, 43 completed the course with good grades. Of these, five changed their majors to become math majors, including the 7-foot tall center on the school's basketball team who dropped basketball because "math was more fun". (They used a computer lab with Derive for all students plus graphing calculators.)

• Judy Fethe of Pellessippi State Technical Community College ran a pilot section of college algebra with 24 "high risk" students few, if any, of whom were expected to pass. However, virtually every student did pass, with surprisingly high grades. The people who staffed the department's tutoring facility came to refer to the class as the "honors section" because the tutors themselves were having so much trouble solving the non-routine problems. (Judy used a computer lab with TEMath plus graphing calculators.) She subsequently followed many of these students into a more traditional trig and precalculus course and found them far more open-minded and willing to try new problems than students who had come through traditional college algebra.

• Joanne Manville of Bunker Hill Community College taught a pilot section in the spring. She reported a high degree of enthusiasm on the part of the students. She also monitored their performance in a traditional calculus course thereafter. She reported that of the four students who took calculus the following summer, all received grades of either A or B. More telling, however, is an incident that especially impressed the calculus instructor. One of the students questioned an answer in the official solutions manual that accompanies the calculus text. The instructor was prepared to find the student's algebraic error, but the student would have none of that -- his problem was that the function, a cubic, could not possibly *behave* the way that the solutions manual claimed!

• Tony Peressini and John Luker of the University of Illinois have given the *Functioning* course for the last three years while coordinating different groups of graduate TA’s who have actually taught most of the sections. They report a significantly higher level of attendance (they're not teaching the same old stuff that the students have seen before). All students have graphing calculators and the format is one of collaborative learning. Tony describes their experiences as

> "The use of this book has raised the intellectual level and utility of our basic algebra course significantly. The students find the course to be very challenging but not overwhelming. They seldom question the usefulness of this content as they frequently did (and probably with good reason) that of the traditional algebra course that we offered prior to adopting this text."

John describes their experience as

> "We chose to use these materials because we wanted something very different from the type of algebra our students had seen in high school. Our surveys indicate that our students prefer this course over the traditional course. At least 1-5 students tell me each semester that this is the first time they have enjoyed a math class. I even had one student tell me he looked forward to doing homework. I was glad I was sitting down at the time."

• Marc Dancer of Deerfield Academy (a high school) gives the *Functioning* course in a totally group work environment with no lecturing at all. An evaluator who sat in on the class one day and interviewed the students reported "They like this class better than other classes because they're able to see how that stuff they learned in algebra fits into real life. They like the strong focus on applications. They like the text for this reason also. They haven't had any tests yet, so they can't comment on their success (as defined by grades), but most feel like they understand the material better than in other math classes. There were two students in the class who had intended this to be their last math class and who are now thinking of going on to calculus in college because of this class." Marc's personal comment on the course is: "This is math for the masses, not just mathematicians." (Each student was required to purchase a TI 82 calculator.)

• In a more quantitative study, Florence Gordon of New York Institute of Technology (one of the co-authors of the text) had the opportunity to teach two parallel sessions of the same precalculus course, one from the *Functioning* materials and the other from a more traditional text to mollify a conservative department chair. She decided to ask some common questions on the two final exams to compare student performance. However, she felt it would be unfair to ask any conceptual or realistic applied problems of the students in the traditional class. Therefore, as the common questions for both groups, she only posed routine (mechanical) questions or posed the routine part of a realistic application to the traditional group. For example, the following question appeared on the exam for the *Functioning* class:

> At a certain pier, the low water line is 6 feet above sea bottom and the high water line is 14 feet above bottom. If low tide occurs at midnight and high tide at 6 am,
> (a) What are the amplitude, frequency and period for this function?
> (b) Sketch the graph, including appropriate scales.
> (c) Write an equation of the water height *H* as a function of time *t*.
> (d) How high is the tide at 11 am?
> (e) When is the water 12 feet deep?

The students in the traditional course were simply asked:

> Let *y* = *f*(*t*) = 10 + 4ßsin[("/6)(*t* - 3)].
> (a') Find the amplitude, frequency and period.
> (d') Find *f*(11).
> (e') 10 + 4ßsin[("/6)(*t* - 3)] = 12. Solve for *t*.

For purposes of comparison, the same number of points were allotted to parts (a) and (a'), (d) and (d'), and (e) and (e'). Out of 11 points, the students in the traditional course scored an average of 3.9 while the students in the *Functioning* course scored an average of 9.6. Based on a small sample *t*- test for the difference of means, this represents a value of *t* = 7.014.

Notice that the students in the *Functioning* course had to understand the context, translate it to a mathematical model, and create the formula that was simply handed to the students in the traditional course. Further, they had to interpret the height of the tide at 11 am as representing H(11). Finally, they had to interpret the meaning of "when is the water 12 feet deep?" as requiring them to set up and solve the equation 10 + 4ßsin["(t - 3)/6] = 12, which was simply handed to the other group.

Incidentally, the students in the *Functioning* course scored better on six of the seven common questions on the two exams, four of them being statistically significant. A full analysis of the results appears in [5].

Working in a realistic context as opposed to an abstract setting appears to make all the difference. The mathematical ideas make sense and the greater mathematical expectations placed on the students becomes acceptable to them. On the other hand, when students in traditional classes are assigned 50 indistinguishable problems every night, all of which look like the same things they think they've seen before, the students are not likely to do many homework problems at all.

• I am currently teaching from the project materials in a college algebra course at Suffolk Community College. Among the numerous positive incidents that occurred, one that stands out is the following. Having covered chapters 1-3 on linear, exponential, power, and logarithmic functions, I was about to introduce polynomials for the first time. To do so, I drew a scatterplot of the prices of a stock that followed a roughly cubic pattern and asked the students to describe the behavior of a function that would model such a pattern. The first response was:

> "*Its concavity changes. First it's concave down and then it's concave up*."

A second student quickly added:

> "*It has just one point of inflection*."

A third student then responded:

"*It changes from increasing to decreasing. First it's increasing, then decreasing, and then increasing again.*"

A fourth student then chimed in with:

"*It doesn't have an inverse*."

And a fifth student observed:

"*It has two turning points.*"

I must admit that this left me standing in front of the group with my mouth agape at a complete loss for words! A few years ago, I would have been thrilled to get a comparable set of responses from students after a full semester of calculus. But in college algebra? And this is not the first time this particular exchange has occurred; last year, I got the first four observations, though in a different order.

Another incident in a different section of the same course, again with students just out of an intermediate algebra course, occurred when the trig ratios were first being introduced, starting with the tangent. I asked the students to use their calculators to find tan 0%, tan 10%, tan 20%, ..., tan 80%. I then raised the natural question about the behavior pattern of this new function based on the data values. Several students immediately called out, simultaneously, comments about it not being linear, or growing faster than linear, because the successive differences were growing. Then three or four students simultaneously called out that it is growing so fast that it must be exponential, to which another student instantly responded: "*No, it's not exponential. I've already checked the successive ratios and it's growing faster than exponential.*"

Perhaps most telling, however, are actual written comments made by students on formal course evaluations. For instance,

"*Math is a part of life -- everything we experience, whether tides or hours of daylight or population growth or rising medical costs, deals with math. I now have a much better understanding of the patterns of life and how math can be applied to them.*"

"My overall reaction to the course was extremely positive. By emphasizing the value of mathematical pursuits through applications first (and theory being derived from the application), the course proved to be constantly interesting. Past math courses seemed tedious. This course never struck me as tedious (challenging? very, but not tedious). Nice to think of math as an intellectual pursuit, a very useful tool, and a way of seeing life -- as opposed to 'a course I have to take to complete a chemistry curriculum'. Above all, being able to see how I could use the subject I was learning in the 'real' world."

Problems to be Faced

In implementing a course such as this, or any reform course for that matter, there are a number of critical challenges that must be faced, depending on the particular institution. One of the most difficult problems involves the support personnel who typically staff a departmental tutoring center. These are usually students (perhaps graduate students) who have been highly successful in traditional courses and who know nothing about the philosophy of a new course or the reasons it is being implemented, who are unfamiliar with the new content, and who will likely try to answer all questions in a very different way from what the instructor might intend.

For instance, several of my students reported the following exchange. They had gone to our tutoring center for help with some problems on fitting a function to data. The tutor working there, a graduate student at a local university, looked at the problems and exclaimed "*You're not supposed to see that until you're in graduate school!*" Or, when some of my students in a Harvard calculus course went in for help on graphical differentiation of a function, the response from several other tutors was: "*But where's the function? Give me the function, I'll differentiate it for you, and then draw its graph.*"

Fortunately, this is a problem that solves itself over the course of several semesters as you build up a cadre of advanced students who have been through the new course or courses. During the first semester or two, however, it can be a major hurdle. One partial solution is to offer a number of workshops for the tutors (and pay them for their time, providing the department or the school has the funds for it) to acquaint them with the changes taking place and to teach them some of the "new" mathematics. Another possibility is to identify the best students in a class and have the school or department pay them to peer tutor their weaker classmates.

A related on-going problem arises at schools that use large numbers of graduate students as TA's in lower level courses. These students have been highly successful in traditional courses and likely have not noticed how many of the students around them simply disappeared. Typically, though, most departments already provide fairly extensive training for the TA's prior to the start of any course and weekly meetings throughout the semester. Implementing a reform course such as the *Functioning* course just makes this more important and likely requires more in the way of preparation. The problem here, though, is that at some institutions, the clear emphasis is on completing one's research and teaching is just a way to earn some money along the way. In such an environment, it very difficult to motivate the TA's to devote the time needed to give justice to such a new course.

A similar problem arises at most other schools that use large numbers of adjunct faculty, who also have been highly successful in traditional math programs. They also need special training before they can teach reform courses. More significantly, such people tend either to hold down full-time jobs elsewhere and the teaching is merely moonlighting or they are teaching as an adjunct at three or four different schools. In either case, there is a natural desire to do things in as simple and fast a way as possible (i.e., the usual) because time is the one commodity they lack. The only advice we can offer is to be highly selective in the choice of which adjuncts are assigned to such a course, picking individuals who have special backgrounds (say who worked as a mathematician in industry) that might make the course special to them, or individuals who have a high energy level who can be turned on by the innovative aspects.

At some schools, perhaps the most severe problem to implementing an innovative course is the resistance of other faculty members. Often, some instructors are philosophically opposed to the innovation because it is not the way that they learned mathematics and how they have been teaching mathematics; some may be intimidated by unfamiliar content; some may fear the loss of topics that they consider important (how many readers remember the Law of Tangents?); some may be uncomfortable with technology; some may be unconvinced about the effectiveness of collaborative learning. Others may simply not want to devote the effort to teaching a new course in a new way.

Some of these objections can be overcome with information. For instance, showing a selection of problems and the level of mathematics, particularly the algebraic tools, needed to solve them, can allay qualms about a course that does not highlight symbolic manipulation. Showing a variety of conceptual problems can convince many people about the mathematical content of an unfamiliar course. Showing some samples of student reports may convince people that there can be a considerable amount of mathematical learning that goes into performing a project and writing the report. Showing information about the changing needs of students and practitioners in other fields may exert some countering pressures regarding the need to change the curriculum. In this regard, for a course such as the *Functioning* course, considerable support can likely be garnered from faculty in other departments because the course content dovetails so well with what they likely would want their students to learn in mathematics. The prevalence of technology throughout our society itself will exert ever greater pressure on faculty to acknowledge that it has a place in the curriculum, though the point should always be made that technology is being used to enhance understanding and learning, not as a substitute or just to provide answers.

Plans for the Future

Although the text for the course has been completed, there remains much to do to affect the mathematics community on a large scale. Most importantly, there will be an on-going need for faculty development and training because there are many innovative features in the *Functioning* course. Much of the mathematical content, such as the notion of fitting functions to data, difference equation models, or even a different slant on traditional topics, may not be familiar to many. For instance, at a recent workshop on collaborative learning at the precalculus level that I ran, I was surprised at the difficulty that experienced faculty were having with the following problem:

> The thermostat in a home is set at 66%. Whenever the temperature drops to 66% (roughly every half-hour), the furnace comes on and stays on until the temperature reaches 70%. Write a trigonometric function that models this situation.

Rather than focusing on the process -- the temperature oscillates up and down between 66 and 70 every 30 minutes -- they focused instead on the mathematical model $y = A + B \sin (Cx + D)$ and tried to adapt it to the situation. Thus, unlike students in the course who let the process guide them and so have little trouble with such a problem, trained mathematicians had considerably more difficulty.

Furthermore, many faculty members are still not comfortable with the use of technology, and most have never tried to implement the use of projects and writing assignments in mathematics or the possible use of collaborative learning. So there is a need to assist in helping such individuals.

In addition, we intend to continue with a variety of dissemination activities, including talks, workshops, and articles, to acquaint mathematics faculty with the course.

Finally, we see the on-going need to continue working in conjunction with the professional organizations such as MAA, AMATYC, and NCTM to promote changes of this type throughout the mathematics curriculum and to continue maintaining contact with other groups and individuals who are likewise working to enhance and revitalize the curriculum at all levels.

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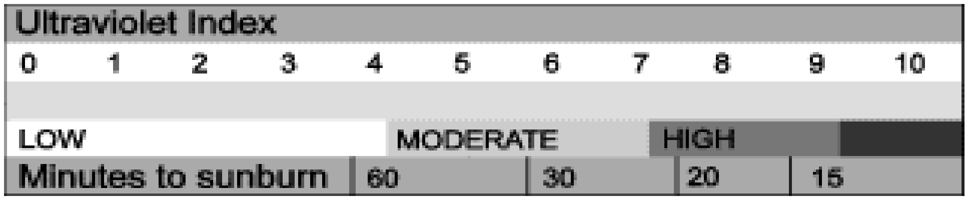
Sample Tests and Final Exam

MA 61 (College Algebra) Test #1

1. The accompanying figure shows the relationship between the UV (ultraviolet) index and the number of minutes it takes to get a sunburn.



a) Is the function linear? Explain your answer.
b) Suppose the UV index is 5. Karen estimates from the table that she will get a sunburn in just over 40 minutes. Using your knowledge of mathematics, is her estimate too high, too low, or about right? Explain your answer.

2. One of the following functions is linear, another is exponential and a third is a power function. (a) Identify which is which. (b) Find the equation of each function.

x	f(x)	x	g(x)	x	h(x)
0	4	1	5	0	4
1	4.8	4	40	1	4.8
2	5.76	7	92.6	2	5.6
3	6.91	10	158.11	3	6.4

3. The 1990 population of California was 29.76 million and growing at an annual rate of 2.3%.
   a) Find an expression for the population at any time t.
   b) What will the population be in the year 2000?
   c) What is the doubling time for the population?
   d) If all things remain the same, when will the population reach 50 million?

4. The annual unemployment rate in the United States over a recent period was:

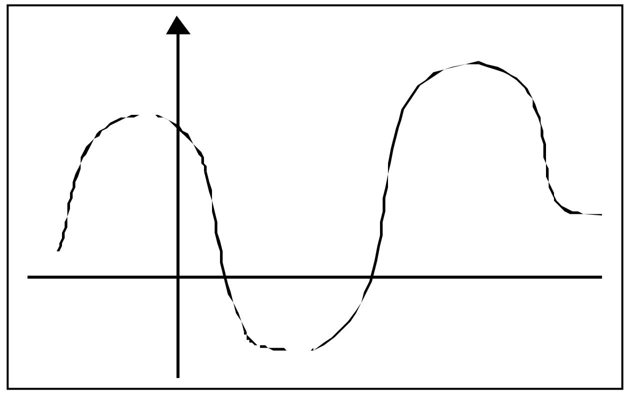
n:	1984	1985	1986	1987	1988	1989	1990	1991
R:	7.5%	7.2	7.0	6.2	5.4	5.3	5.5	6.2

a) Use graph paper to draw the scatterplot for this data and sketch the best-fit line by eye.
b) Estimate the slope of this line and tell what it means.
c) What is your best estimate for the equation of this line?
d) Using this line, what is your estimate for the unemployment rate in 1995?

5. Sketch the graph of a function which is increasing and concave down from point A to point B. Let M be the midpoint on this curve. The three points A, B and M determine three line segments, AM, MB and AB. List them in the order of *increasing* slope. Would you answer change if the curve were increasing and concave up?

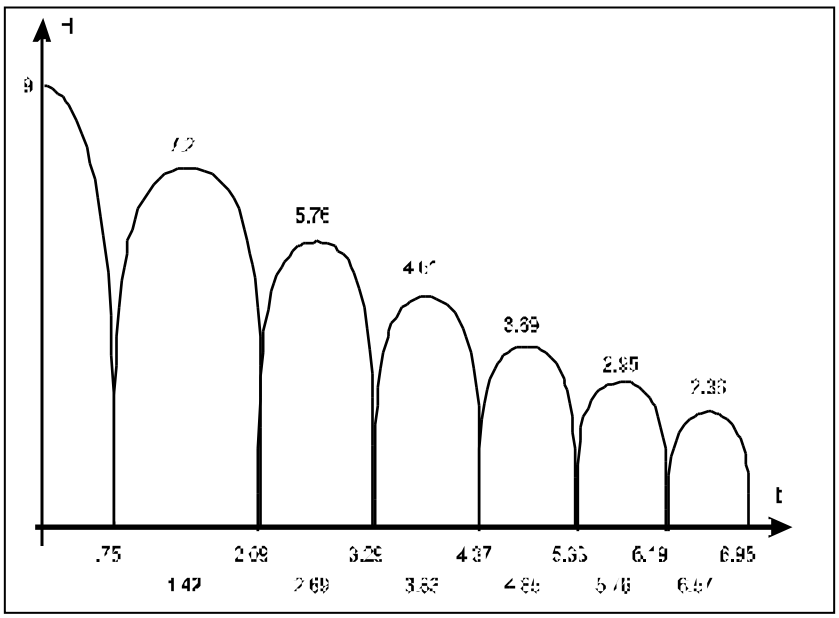
6. Sketch the graph of a <u>single</u> smooth function that has **all** of the following properties:
   a) $f(0) = 6$
   b) f(x) is decreasing for $0 \le x \le 3$
   c) f(x) is increasing for $3 < x \le 5$
   d) f(x) is decreasing for $x > 5$
   e) f(x) approaches 1 as x approaches ñ.

7. For the function shown, indicate:
a) all intervals where it is increasing.
b) all intervals where it is decreasing.
c) all intervals where it is concave up.
d) all intervals where it is concave down.
e) all points of inflection.



MA 61 (College Algebra) Test #2

1. An experiment is conducted in which a ball is dropped from an initial height of 9 feet and its height above floor level as a function of time is recorded and displayed, as in the figure shown. When the curve is traced out, the measurements indicated on the graph are obtained for the times when the ball hits the floor, the times when the ball reaches its maximum heights, and the values of the maximum heights.



n	Time		n	Max ht
1	.75		1	9
2	2.09		2	7.2
3	3.29		3	5.76
4	4.37		4	4.61
5	5.33		5	3.69
6	6.19		6	2.95
7	6.95		7	2.36

(a) Notice that the times when the ball hits the floor appear to follow a linear pattern. Use your calculator to find the best linear fit to these times as a function of the number of the bounce; that is, bounce number $n = 1$ occurs at time $t = .75$, etc.

(b) Draw the scatterplot for the maximum heights.

(c) Notice that the maximum heights do not follow a linear pattern as a function of n. Explain why you would expect this pattern to be exponential rather than a power function.

(d) How would you transform this data to linearize it?

(e) Use your calculator to find the equation of the line that best fits this transformed data.

(f) Undo the transformation to find the best exponential function that fits these data values as a function of n.

(g) What is the practical meaning of the base you obtain?

(h) Use the results you obtained above to predict the next time the ball will hit the ground and the maximum height to which it will rise on the next bounce of the ball.

2. Determine which of the following functions have an inverse. For any that do, find $f^{-1}(7)$.

a) $f(x) = x^2 + 3^x$ b) $f(x) = x^3 + 3^x$

3. Suppose you are told that the *real* roots of a polynomial are x = 1, 4, -3, 2.5 and -1.5.

a) What is the minimum degree of the polynomial?

b) Assuming there are no complex roots, write a possible formula for this polynomial. (Do not expand it.)

c) Draw two possible graphs for this polynomial.

d) Indicate roughly where the function has turning points.

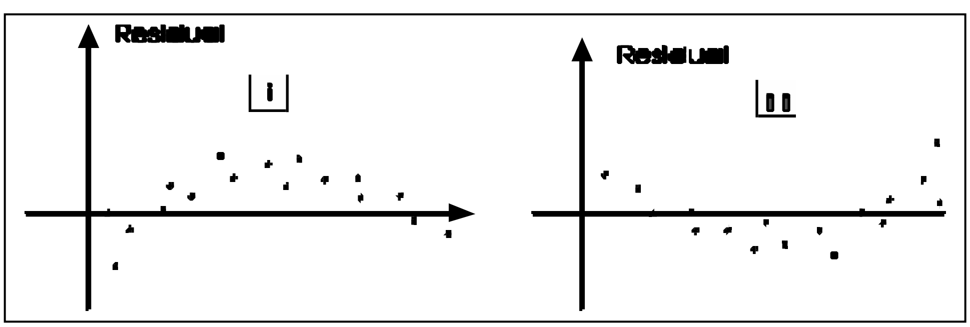
e) Indicate roughly where the function has inflection points.

4. A quadratic polynomial has a real root at $x = 2$ and a turning point at (1,5). Find the equation of the quadratic.

5. The best fit line is constructed for each of four sets of nonlinear data. Their patterns are roughly:

(a) increasing and concave up; (b) increasing and concave down;
© decreasing and concave up; (d) decreasing and concave down.

Match each with one of the possible residual plots shown below. Explain your answer in each case.



MA 62 (Precalculus) **Test #2**

1. Find the complete solution of each of the following difference equations:

a) $x_{n+1} = 1.2\ x_n$ b) $x_{n+1} = \_\ x_n$ c) $x_{n+1} = -2\ x_n$

Assuming that the initial value $x_0 = 1$ in each of the above solutions, draw a *very rough* sketch showing the behavior of the solution.

2. The "half-life" of aspirin in the body is 30 minutes. If a person takes 1000 mg (two tablets), what is the effective dosage after 2_ hours?

3. A can of soda at 70% is put into the freezer kept at a constant temperature of 10% to chill. Suppose that the temperature of the can is 55% after 20 minutes and that its temperature is 45% after 30 minutes. Sketch the graph of the temperature as a function of time and use the concavity of the graph to answer the following: Which of the following temperature readings are possible and which are impossible?

a) T(25) = 52% b) T(25) = 48% c) T(35) = 42% d) T(35) = 38%

Give reasons for each of your answers.

4. The population of a certain country was 80 million in 1990 and is growing at an annual rate of 2%.

a) Write a *difference equation* for the population P_n in this country after n years.

b) Further, starting in 1990, the country agreed to allow in 1 million immigrants each year. Write a difference equation for the population P_n in this country after n years.

5. The number of *new* cases of a certain disease each year has been dropping 10% per year since 1950. If there were 8000 new cases in 1960, what is the total number of new cases between 1960 and 1994?

6. A population grows according to the logistic difference equation $P_{n+1} = 1.04P_n - .00005P_n^2$. (Equivalently, $P_n = .04P_n - .00005P_n^2$) Sketch a graph of the behavior of this population, paying careful attention to concavity, if:

a) $P_0 = 100$ b) $P_0 = 600$ c) $P_0 = 1000$

7. Consider the difference equation $x_{n+1} = \_x_n + 2n$, $x_0 = 40$. Calculate the next three terms x_1, x_2 and x_3 of the solution sequence from the difference equation and tell whether the solution appears to be increasing or decreasing, concave up or concave down.

8. Suppose you are told that the *real* roots of a polynomial are x = 1, 4, -3, 2.5 and -1.5.

a) What is the minimum degree of the polynomial?

b) Assuming there are no complex roots, write a possible formula for this polynomial. (Do not expand it.)

c) Draw two possible graphs for this polynomial.

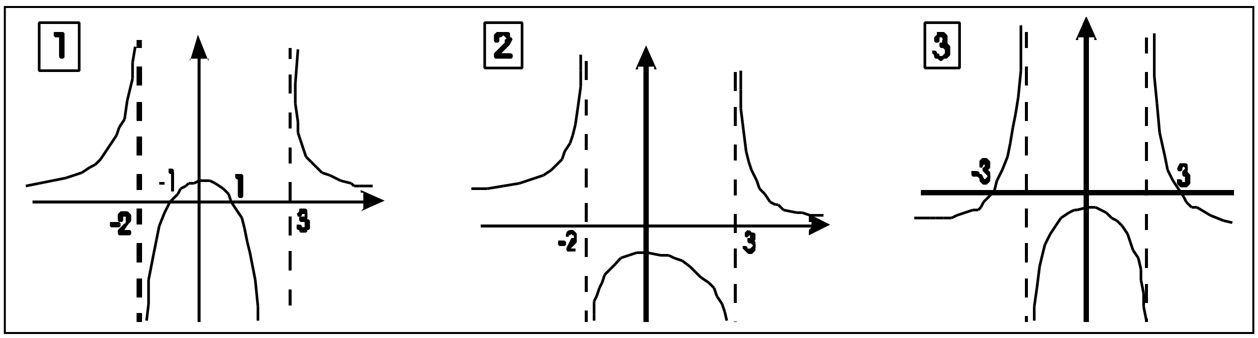
9. Perform the first two iterations of the bisection method to estimate the first positive root of $f(x) = x^3 - 4x + 1 = 0$. Use your calculator to locate any other real roots correct to three decimal places.

MA 61 (College Algebra) Test #3

1. Two functions *f* and *g* are defined in the following table. Use the given values to complete the table:

x	$f(x)$	$g(x)$	$f(x)$ - $g(x)$	$f(x)$ß$g(x)$	$f(g(x))$	$g(f(x))$
0	1	0				
1	2	3				
2	3	1				
3	0	2				

2. Match each of the following functions with its graph:
(a) $y = (x^2 - 1)/(x^2 - x - 6)$ (b) $y = (x^2 + 1)/(x^2 - x - 6)$ © $y = (9 - x^2)/(x^2 - 4)$

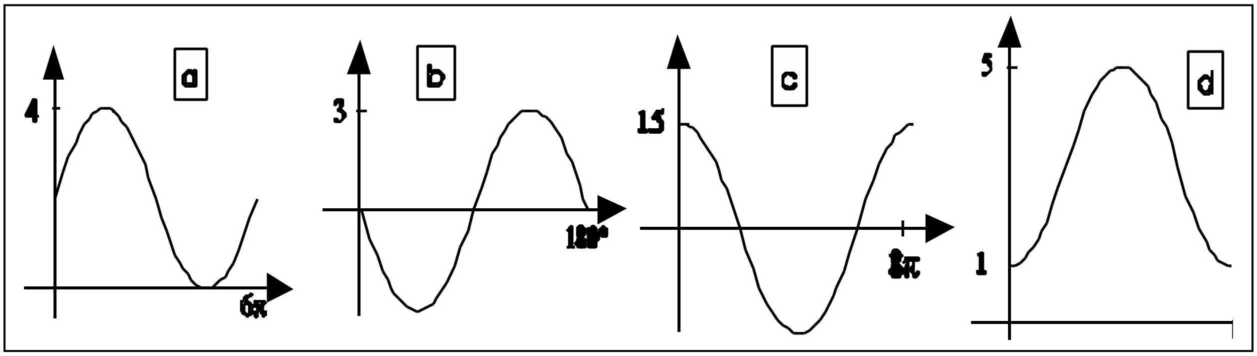


3. The thermostat in a house is set for 66%. When the temperature drops to 66%, the heat goes on and stays on until the temperature reaches 70%, when it goes off. The cycle takes 30 minutes.
a) Sketch the graph of the temperature in the house as a function of time.
b) Construct a sinusoidal function that models the temperature.
c) What are the amplitude, vertical shift, period, and frequency of this function?
d) What will the temperature reading be after 5 minutes?
e) Use your calculator to estimate how long it will take for the temperature to rise to 69%.

4. Suppose that you dropped your calculator so that the SIN and TAN keys are broken and it is stuck in radian mode.
a) Convert 40% into radians and find cosine of that angle.
b) Find the sine and tangent of that angle using the broken calculator.

5. When the space shuttle comes in for a landing at Cape Kennedy, it descends from a height of 10,000 feet at an angle of 19% with the horizontal.
(a) What actual distance does the shuttle traverse along this final glide path?
(b) How far down range should the shuttle be when it passes the 10,000 foot altitude?

6. Identify the following trigonometric functions from their graphs:



MA 61 (College Algebra) Final Exam

1. You are riding a roller coaster at an amusement park. You must climb a flight of stairs to get to the starting platform for the ride. The graph below shows your height as a function of time during the ride.

a) Indicate all intervals where the function is positive. What do they represent?
b) Indicate all intervals where the function is increasing. What do they represent?
c) Indicate all intervals where the function is decreasing. What do they represent?
d) Indicate all intervals where the curve is concave up. What do they represent?
e) Indicate all intervals where the curve is concave down.
f) Indicate all points of inflection.

2. A biologist is attempting to develop a mathematical model relating a person's height (in inches) to his or her weight (in pounds) from infancy to maturity. She suspects the relationship involves a *power function* law. The data values for one person studied are:

Wt:	5	17	24	33	42	59	83	98	120	195
Ht:	19	26	31	40	45	52	57	64	68	75

a) Draw the scatterplot for this data and indicate the regression line by eye.
b) Transform this data to linearize it, draw the scatterplot for the transformed data and indicate its regression line.
c) Estimate the equation for the regression line you drew in part b).

Suppose the computer produces the following linear regression equation for the transformed data:

$$\mathbf{Y} = .416\,\mathbf{X} + .957$$

d) De-transform this equation to produce the power function which best fits the data.
e) What is your prediction for the person's height when he weighed 150 pounds?
f) What is your prediction for the person's height if he weighed 210 pounds?
g) Which prediction would you have more confidence in? Why?

3. A can of warm soda (80%) is placed into the refrigerator at 40% to cool. After 15 minutes, the temperature of the soda is 60%.

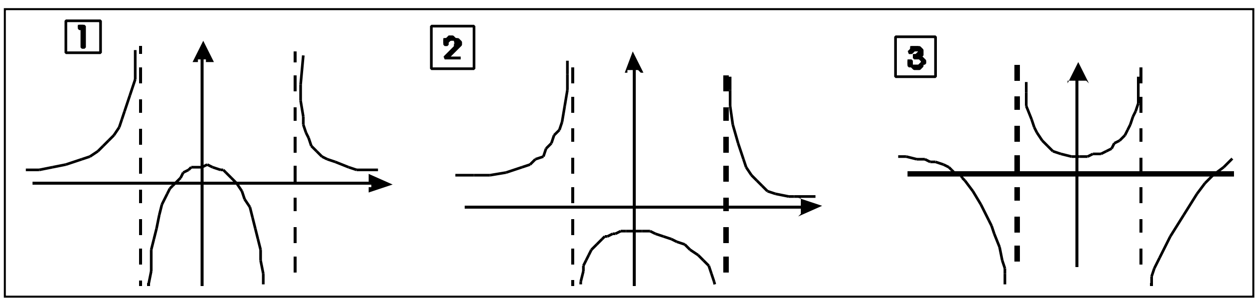
a) Assuming, incorrectly, that the temperature drops linearly, find the equation of the linear function and use it to predict how long it will take for the temperature of the soda to reach 45%.
b) Assuming that the temperature drops exponentially, find the equation of the exponential function that models the temperature of the soda and use it to predict how long it will take for the temperature to reach 45%

4. Two functions f and g are defined in the following table. Use the given values to complete the table. (Note: Some of the values may be undefined; if so, mark them as such in the table.)

x	$f(x)$	$g(x)$	$f(x + 1)$	$g(2x)$	$g(f(x))$	$g(f(x))$	$f^{-1}(x)$
1	2	4					
2	4	1					
3	3	2					
4	1	3					

5. Suppose you are told that the *real* roots of a polynomial are x = 1, 3, 6, -3, -.5 and -1.5.
   a) What is the minimum degree of the polynomial?
   b) If there are no complex roots, write a possible formula for this polynomial.
   c) Draw two possible graphs for this polynomial.
   d) Indicate roughly where the function has turning points.
   e) Indicate roughly where the function has inflection points.

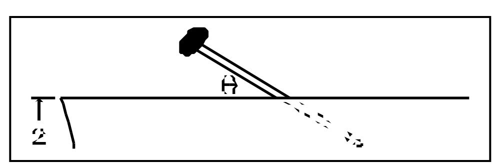
6. Match each of the following functions with its graph and label all roots and asymptotes on the graphs.
(a) $y = (x^2 - x - 6)/(x^2 - x - 2)$ (b) $y = (x^2 - x - 2)/(x^2 - x - 6)$ © $y = (x^2 - x + 2)/(x^2 - x - 6)$



7. The temperature in an oven cycles on and off every 20 minutes. When the temperature reaches 330%, the heat goes off and stays off until the temperature reaches 312%, when it goes back on.
  a) Sketch the graph of the temperature in the oven as a function of time.
  b) Construct a sinusoidal function that models the temperature.
  c) What are the amplitude, vertical shift, period, and frequency of this function?
  d) What will the temperature reading be after 3 minutes?
  e) Use your calculator to estimate how long it will take for the temperature to rise to 325%.

8. Suppose that you dropped your calculator so that the SIN and TAN keys are broken and it is stuck in radian mode.
  a) Convert .24 radians into degrees and find cosine of that angle.
  b) Find the sine and tangent of that angle using the broken calculator.
  c) Use an appropriate double angle formula to find the sine of .48 radians.

9. You will be hammering a three inch-long nail into a piece of wood that is two inches thick. Find the largest angle at which you can hammer the nail all the way into the wood without it coming out the opposite side.



10. You are about to slice up cranberry sauce to go with your holiday turkey. The cranberry sauce comes out of a can and has a diameter of 3 inches. When you slice the roll of cranberry sauce at an

angle, each slice will be an ellipse; the minor axis will be a constant 3 inches. Suppose that you slice the roll at an angle of 27% to the vertical. Find the length of the major axis of each slice.

11. Write a possible formula for each of the following trigonometric functions: