Unexpected Occurrences of the Number *e*

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If numbers are randomly chosen from the interval [0,1] until their sum exceeds 1, the expected number of selections is e.

If numbers are randomly chosen from the interval [0,1] until a number is selected that is less than its predecessor, the expected number of selections is e.

If a group of people check their hats and later have them randomly returned, the probability that no one winds up with his own hat is approximately $1/e$.

If a company gives away one of n coins with each purchase, then after n purchases the fraction of different coins one can expect to have is approximately $1 - 1/e$.

An employer must choose or reject an applicant at the time of the interview. In order to maximize the probability of choosing the best of the applicants, the employer should interview approximately $1/e$ of the applicants and then choose the first one thereafter who is superior to all of those; in this case the probability of choosing the best applicant is approximately $1/e$.

Suppose n cars are parked in a line and ours is in one of these n equally likely positions. Assuming we cannot drive out until the drivers in front of us have arrived and assuming these drivers arrive in a random order, the expected number of drivers that need to arrive until we are clear to leave is approximately $n - 1 - \ln \frac{n + 0.5}{1.5}$.

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Review of Expected Value

Ten cards numbered 1, 2, . . ., 10

Choose one card:

Card	Payoff
Ten	\$25
Prime	\$6
Other	\$0

Play the game ten times.

“Expect” each number once.

Expected total payoffs

$= 25 \times 1 + 6 \times 4 + 0 \times 5 = \$49$

Expected payoff per game = \$4.90

Expected payoff per game

$$= 25\times\frac{1}{10} + 6\times\frac{4}{10} + 0\times\frac{5}{10} = \$4.90$$

Expected Value: Multiply each possible value by its associated probability. Sum the resulting products.

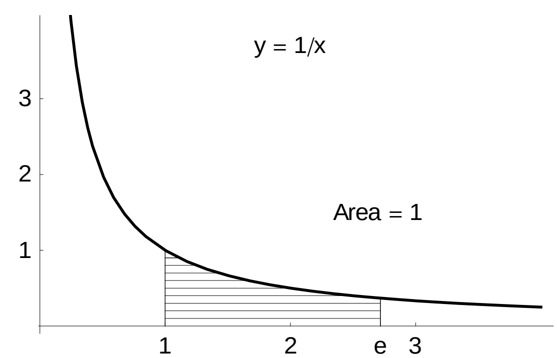
Example: Toss a coin three times. What is the expected number of heads?

Number of heads	0	1	2	3
Probability	1/8	3/8	3/8	1/8

Expected number of heads

$$= 0 + \frac{3}{8} + \frac{6}{8} + \frac{3}{8} = 1.5$$

Review of e



$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

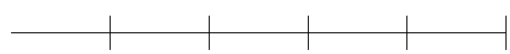
$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$$

Randomly choose numbers a, b, c, . . .

from $\{x \mid 0 \le x \le 1\}$.

What is the expected number of choices necessary until the sum exceeds 1?

Let’s approximate the answer by simplifying the problem: Randomly choose (with replacement) numbers a, b, c, . . . from the set {1/5, 2/5, 3/5, 4/5, 5/5}.



Let N = number of choices necessary until the sum exceeds 1.

Clearly, $P(N > 1) = 1$.

$N > 2$ means $a + b \le 1$. Thus,

$a,\ a + b \in \{1/5, 2/5, 3/5, 4/5, 5/5\}$: $\binom{5}{2}$ ways.

$$P(N>2) = \frac{\binom{5}{2}}{5^2} = 10\left(\frac{1}{5}\right)^2$$

N > 3 means a + b + c $\leq$ 1. Thus,

a, a + b, a + b + c $\in$ {1/5,2/5,3/5,4/5,5/5}: $\binom{5}{3}$ ways.

$$P(N>3) = \frac{\binom{5}{3}}{5^3} = 10\left(\frac{1}{5}\right)^3$$

$$P(N>4) = \frac{\binom{5}{4}}{5^4} = 5\left(\frac{1}{5}\right)^4$$

$$P(N>5) = \left(\frac{1}{5}\right)^5$$

$$E(N) = 2P_2 + 3P_3 + 4P_4 + 5P_5 + 6P_6$$

$$\begin{aligned} E(N) = {} & P_2 + P_3 + P_4 + P_5 + P_6 \\ & + P_2 + P_3 + P_4 + P_5 + P_6 \\ & + P_3 + P_4 + P_5 + P_6 \\ & + P_4 + P_5 + P_6 \\ & + P_5 + P_6 \\ & + P_6 \end{aligned}$$

$$E(N) = 1 + 1 + P(N > 2) + P(N > 3) + P(N > 4) + P(N > 5)$$

$$E(N)=1+5\left(\frac{1}{5}\right)+10\left(\frac{1}{5}\right)^2+10\left(\frac{1}{5}\right)^3+5\left(\frac{1}{5}\right)^4+\left(\frac{1}{5}\right)^5$$

$$E(N)=\left(1+\frac{1}{5}\right)^5 \approx 2.5$$

Generalize: Randomly choose (with replacement) numbers a, b, c, . . . from the set {1/n, 2/n, 3/n, . . ., n/n}.

The expected number of choices necessary until the sum exceeds 1 is

$$\left(1+\frac{1}{n}\right)^n \to e \approx 2.71828$$

Randomly choose numbers a, b, c, . . .

from {x | 0 $\leq$ x $\leq$ 1}. The expected number of choices necessary until the sum exceeds 1 is e.

Randomly choose numbers a, b, c, . . .

from {x | 0 $\leq$ x $\leq$ 1} as long as the sequence is increasing. What is the expected length L of the sequence (including the first number that reverses the increasing direction)?

<u>Example:</u> 0.15, 0.33, 0.58, 0.47 (L = 4)

<u>Example:</u> 0.68, 0.24 (L = 2)

$P(L > 1) = 1$

$P(L > 2) = P(b > a) = \frac{1}{2}$

$P(L > 3) = P(c > b > a) = \frac{1}{3!}$

$P(L > 4) = P(d > c > b > a) = \frac{1}{4!}$

$P(L > n) = \frac{1}{n!}$

$$E(L) = 2P_2 + 3P_3 + 4P_4 + 5P_5 + \ldots$$

$$\begin{aligned} E(L) = {} & P_2 + P_3 + P_4 + P_5 + \ldots \\ & + P_2 + P_3 + P_4 + P_5 + \ldots \\ & \qquad + P_3 + P_4 + P_5 + \ldots \\ & \qquad\qquad + P_4 + P_5 + \ldots \\ & \qquad\qquad\qquad + P_5 + \ldots \end{aligned}$$

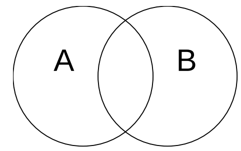
$$E(L) = 1 + 1 + P(L > 2) + P(L > 3) + \ldots$$

But, $P(L > n) = \frac{1}{n!}$

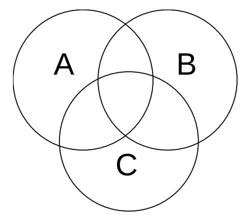
$$E(L) = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots$$

$$E(L) = e$$

If a group of people check their hats and later have them randomly returned, what is the probability that no one winds up with his own hat?



$$n(A \cup B) = n(A) + n(B) - n(AB)$$



$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - [n(AB) + n(AC) + n(BC)] + n(ABC)$$

Suppose 6 people check their hats.
Let U = set of all permutations of $\{1, 2, 3, 4, 5, 6\}$
$n(U) = 6!$
Let A_i = set of all permutations of $\{1, 2, 3, 4, 5, 6\}$ for which $i \to i$.
$n(A_i) = 5!$
$n(A_iA_j) = 4!$
$n(A_iA_jA_k) = 3!$, etc.

$$\begin{aligned} & n(A_1 \cup A_2 \cup \cdots \cup A_6) \\ & = (6)(5!) - \binom{6}{2}(4!) + \binom{6}{3}(3!) \\ & - \binom{6}{4}(2!) + \binom{6}{5}(1!) - \binom{6}{6}(0!) \end{aligned}$$

Probability at least one gets his own hat back is

$$\frac{6! - \binom{6}{2}(4!) + \binom{6}{3}(3!) - \binom{6}{4}(2!) + \binom{6}{5}(1!) - \binom{6}{6}(0!)}{6!}$$

Probability at least one gets his own hat back is

$$\frac{6! - \frac{6!}{2!4!}4! + \frac{6!}{3!3!}3! - \frac{6!}{4!2!}2! + \frac{6!}{5!1!}1! - \frac{6!}{6!0!}(0!)}{6!}$$

Probability at least one gets his own hat back is

$$1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \frac{1}{5!} - \frac{1}{6!}$$

Probability no one gets his own hat back is

$$1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!}$$

If n people check their hats, then the probability no one gets his own hat back is

$$1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \ldots + (-1)^n \frac{1}{n!} \approx e^{-1} \text{ or } \frac{1}{e} \approx 0.368$$

Recall, $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

MacBurgers gives out coins commemorating each of the 50 states. How many different coins can we expect after 50 purchases?

$$P(5^{th}\text{ coin is new}) = \left(\frac{49}{50}\right)^4$$

Expected number of new coins on the 5^{th} day is

$$0\times P(5^{th}\text{ is not new}) + 1\times P(5^{th}\text{ is new})$$

$$= \left(\frac{49}{50}\right)^4$$

Slide 13

Expected number of new coins on the k^{th} day is

$$\left(\frac{49}{50}\right)^{k-1}$$

After 50 days we can expect

$$\sum_{k=1}^{50}\left(\frac{49}{50}\right)^{k-1} = \frac{1-\left(\frac{49}{50}\right)^{50}}{1-\frac{49}{50}}$$

$$= 50\left[1-\left(\frac{49}{50}\right)^{50}\right] \approx 31.79$$

If there are n different coins, then after n days we can expect $E = n[1 - (1 - \frac{1}{n})^n]$

$$\frac{E}{n} = 1-\left(1-\frac{1}{n}\right)^n \approx 1-\frac{1}{e} \approx 0.6321$$

Slide 14

An employer must choose or reject an applicant at the time of the interview. The employer plans to interview a certain number k of n applicants and then choose the first one thereafter who is superior to all of the first k.

Let p_k = the probability of getting the best secretary by selecting the first applicant who is superior to all of the first k.

<u>Example:</u> With n = 7, what is p_3?

$$P_3 = \frac{1}{7} + \frac{1}{7}\times\frac{3}{4} + \frac{1}{7}\times\frac{3}{5} + \frac{1}{7}\times\frac{3}{6}$$

$$P_3 = \frac{3}{7}\left(\frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6}\right) = 0.407$$

Slide 15

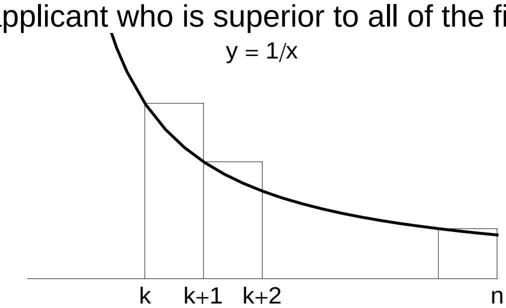
$$P_3 = \frac{3}{7}\left(\frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6}\right) = 0.407$$

$$P_2 = \frac{2}{7}\left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6}\right) = 0.414$$

In general, with n candidates,

$$P_k = \frac{k}{n}\left(\frac{1}{k} + \frac{1}{k+1} + \cdots + \frac{1}{n-1}\right),$$

where p_k is the probability of getting the best secretary by selecting the first applicant who is superior to all of the first k.



Slide 16

$$\frac{1}{k} + \frac{1}{k+1} + \cdots + \frac{1}{n-1} \approx \int_k^n \frac{1}{x}\,dx = \ln n - \ln k$$

So, $P_k \approx \frac{k}{n}\left(\ln n - \ln k\right) = \frac{k}{n} \ln \frac{n}{k}$.

$$\text{Or, } P_k \approx -\frac{k}{n} \ln \frac{k}{n}$$

What is the maximum of F(x) = –x ln x?

$$F'(x) = -\ln x - 1$$

$$F'\left(\frac{1}{e}\right) = 0 \text{ and } F\left(\frac{1}{e}\right) = \frac{1}{e}$$

The employer should interview approximately 1/e of the applicants and then choose the first one thereafter who is superior to all of those; in this case the probability of choosing the best applicant is approximately 1/e .

<u>Hollywood Bowl Parking:</u> Suppose *n* cars are parked in a line and ours is in one of these n equally likely positions. Assuming we cannot drive out until the drivers in front of us have arrived and assuming these drivers arrive in a random order, the expected number of drivers that need to arrive until we are clear to leave is approximately

$$n - 1 - \ln \frac{n + 0.5}{1.5} .$$