

Reflective Abstraction And The Concept Of Limit: A Quasi-Experimental Study To Improve Student Performance In College Calculus By Promoting Reflective Abstraction Through Individual, Peer, Instructor And Curriculum Initiatives

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Why Calculus?

- Dropout and failure rates in calculus are excessively high. Even those who pass perform poorly on calculus skills and concepts (Cipra 1988 [1]; Peterson 1986 [2]).
- Selden, Selden and Mason (1994) [3] demonstrate that the best calculus students are unable to successfully complete non-routine calculus problems.
- Epp (1987) [4] writes, “The fact is that the state of most students’ conceptual knowledge of mathematics after they have taken a calculus course is abysmal” (p. 48).
- Committee on the Undergraduate Program in Mathematics (2004) [5] claims that the total number of mathematics majors is decreasing and the enrollment in advanced mathematics classes is also declining.

Why The Limit?

- “Although the function concept is central to modern mathematics, it is the concept of a limit that signifies a move to a higher plane of mathematical thinking” (Tall, 1992, p. 501 [6]).
- Cornu (1991) [7] believes that the limit is typically the first mathematical concept that students encounter where one cannot get a solution with a direct computation. He claims that limits are “surrounded with mystery . . . one must arrive at one’s destination by a circuitous route” (p. 151).
- Teachers with several years experience teaching calculus also struggle with the concept of limit (Simonsen, 1995) [8].

•Cornu (1991) [7] writes, “The mathematical concept of a limit is a particularly difficult notion, typical of the kind of thought required in advanced mathematics. It holds a central position, which permeates the whole of mathematical analysis – as a foundation of the theory of approximation, of continuity, and of differential and integral calculus” (p. 153).

How to Improve Student Understanding of the Limit?

•Initiate Reflective Abstraction

Research Question

Can a curriculum that initiates reflective abstraction improve student performance on the concept of limit?

RUMEC Definition

Reflective abstraction is a concept introduced by Piaget to describe the construction of logico-mathematical structures by an individual during the course of cognitive development. Reflective abstraction by an individual proceeds from two mechanisms which are necessarily associated. They are projection unto a higher level of that which was derived from a lower level, and secondly reflection, which reconstructs and reorganizes within a larger system what is transferred by projection (DeVries, 2001)[9].

Cooley Definition

Reflective abstraction is a mechanism for the isolation of particular attributes of a mathematical structure that allows the subject to construct or reconstruct knowledge that is new, that is, knowledge not previously known. A feature of reflective abstraction is that it clarifies and organizes logico-mathematical experiences in such a way as to recognize both nuances and broad generalizations among them. Any new constructions will be associated with knowledge the subject already has. The subject orders or re-orders a class of situations with the characteristics of the current object so that the new knowledge fits with previous schemas, or the previous schema has been reconstructed. The new generalization occurs precisely because of a mental construction or reconstruction (Cooley, 2002, p. 255) [10].

Constructs of Reflective Abstraction (Dubinsky & Lewin, 1986 [11], and Dubinsky, 1991 [12])

- Interiorization
- Coordination
- Generalization
- Encapsulation
- Reversal

Working Definitions

Interiorization

•A student performs the steps in a procedure, The student reflects on the procedure and begins to define a concept.

Coordination

•A student examines two different processes and integrates them into a coordinated process that is used to analyze a mathematical concept.

Encapsulation

•A student encapsulates a concept by constructing individual meaning. Encapsulation is the act of personifying a concept. An abstract notion or a collection of abstract notions becomes meaningful to an individual.

Generalization

•After an individual has encapsulated a notion, it is extended and applied to a wider collection of mathematical problems.

Reversal

•A student constructs a new mathematical notion by reversing the steps of the original notion.

Initiates

- Individual
- Peer
- Instructor
- Curriculum

Experimental Design

•Quantitative Analysis
–Which section of students performed better?

•Qualitative Analysis

–How were the experiences in the two sections similar and how were they different?
–What are the characteristics of successful students and how do these differ from the characteristics of less successful students?

Why This Design?

- The quantitative model reflects the standards of assessment used at the community college.
- The quantitative analysis establishes correlation between reflective abstraction and student performance on the limit.
- The qualitative analysis provides evidence of causation between reflective abstraction and student performance on the limit.

Experimental Design

Two sections of Calculus I students participated in the study.

One section studied an experimental curriculum designed to initiate reflective abstraction.

One section studied a traditional curriculum.

Pretest-Posttest Model

- Students completed a pretest and a posttest on the concept of limit.

- These were scored using a 0, 1, 2 rubric from Illinois State Board of Education [13].

- Scores were coordinated with an independent grader.

Pretest Results

Scores on Pretest – All Participating Students

	<u>Experimental</u>	<u>Control</u>
Sample size	16	19
Mean	7.313	6.316
St. Dev.	5.237	5.132

$t(32) = 0.57, p = .58$, No significant difference was identified.

Posttest Results

Scores on Posttest – All Participating Students

	<u>Experimental</u>	<u>Control</u>
Sample size	16	19
Mean	21.18	16.95
St. Dev.	4.48	5.14

$t(32) = 2.63, p < .01$.

A significant difference favoring the experimental section was identified.

Scores on Posttest Restricted to Students Who Attended All Five Classes

	Experimental	Control
Sample size	13	15
Mean	21.38	18.40
St. Dev.	4.50	4.47

$t(25) = 1.76, p = .046.$

A significant difference favoring the experimental section was identified.

Analysis of Variance for Post, using Adjusted SS for Tests

Source	<i>df</i>	Seq SS	Adj SS	Adj MS	<i>F</i>	<i>p</i>
Pretest	1	148.25	121.03	121.03	6.01	0.020
SectionType	1	128.94	128.94	128.94	6.40	0.017
Error	32	644.36	644.36	20.14		
Total	34	921.54				

Summary of Quantitative Analysis

- No significant difference was identified for the pretests.
- The students in the experimental section scored significantly higher on the posttest than the students in the control section.
- When posttest scores are covaried against pretest scores, a significant difference is identified.

Qualitative Analysis

- Based on results from the pretest and posttest, twelve students were chosen to participate in the qualitative analysis.

- 2 students with highest improvement from each section were selected.

- 2 students with median improvement from each section were selected.

- 2 students with least improvement from each section.

Extended Response Rubric

- The twelve students in the comparison subgroup had their posttest re-scored using the five-point extended response rubric from Illinois State Board of Education [14]. The rubric measures the following:

- Mathematical Knowledge

- Strategic Knowledge

- Explanation

Mathematical Knowledge

	Experimental	Control
N	6	6
Mean	40.67	37.17
St. Dev.	5.20	5.81

$t(10) = 1.10.$

$p = .15$

No significant difference was identified.

Strategic Knowledge

	Experimental	Control
N	6	6
Mean	41.50	37.17
St. Dev.	6.28	5.11

$t(10) = 1.31.$

$p = .11$

No significant difference was identified.

Explanation		
	Experimental	Control
N	6	6
Mean	44.33	35.50
St. Dev.	7.09	8.31

$t(10) = 1.98.$

$p = .04$

A significant difference favoring the experimental section was identified.

Classification Scheme

- Improve Category

–six students who scored below the median on the pretest and at, or above the median, on the posttest

- Maintain Category

–two students who scored above the median on both the pretest and the posttest

- Regress Category

–four students who scored above the median on the pretest but scored at the median level or below on the posttest

Interviews

- Each of the twelve students was interviewed. These interviews were designed to clarify student understanding on the posttests.

- The interviews were transcribed and coded based on the demonstrated constructs of reflective abstraction.

Number of Examples of Constructs of Reflective Abstraction – Improve Group

Name	I	C	G	R	E
Dan	5	10	5	4	1
Sam	8	13	2	5	1
Ron	2	15	2	4	0
Bruce	7	9	6	6	0
Chuck	9	6	1	2	0
Karen	9	10	4	2	0
Totals	40	63	20	23	2

Number of Examples of Constructs of Reflective Abstraction – Maintain Group

Name	I	C	G	R	E
Larry	6	16	2	1	1
Maria	3	14	0	3	0
Totals	9	30	2	4	1

Number of Examples of Constructs of Reflective Abstraction – Regress Group

Name	I	C	G	R	E
Greg	6	7	1	2	0
John	4	8	0	4	1
Vern	6	6	2	5	0
Alice	3	15	1	2	0
Totals	19	36	4	13	1

Summary of Interview Analysis

1. Successful students engage in reflective abstraction more often than less successful students.

Improve: 24.67 cases per student.

Maintain: 23.00 cases per student.

Regress: 18.25 cases per student.

2. Coordination appeared most often.
3. Improve group students demonstrated generalization much more often than the other students.
4. Encapsulation appeared rarely.

Problems with Encapsulation

Dubinsky (1986) writes, “It is only later (and it may not happen for everyone) that the epistemic subject sees the operation as a total structure. Reflective abstraction includes the act of reflecting on one’s cognitive action and coming to perceive the collection of thoughts as a structured whole. As a result, the subject can now encapsulate the structure, and see it as an alimnt for other structures” (p. 63).

Why no Encapsulation?

- Short time frame of five days may not be enough to see encapsulation.
- Difficult to determine if a student has truly encapsulated a notion or is simply mimicking the terminology.

One Possible Example of Encapsulation

•A limit is when, as the function is getting closer and closer to the same x -value from the left and the right, the function is getting closer and closer to the same y -value from the left and from the right. Continuous graphs always have a limit for any x -value. You can draw a continuous graph without lifting your pencil from the paper. A graph that is not continuous, you have to lift your pencil from the paper to keep drawing it. Functions that have a limit even though they are not continuous reach the same number from the left and from the right even if there may or may not be a y -value. Examples include functions with removable discontinuities that create holes in graphs. You can use the all-but-one-point rule to find the limit. A function that is not continuous and has no limit is because the function gets closer to a different number from the left and from the right. You can use tables or graphs to see this.

How Did the Experimental Curriculum Promote Reflective Abstraction?

The experimental curriculum was designed to promote reflective abstraction through

1. Individual Initiates
2. Peer Initiates
3. Curriculum Initiates
4. Instructor Initiates

Individual Initiates

Students in the experimental section completed text book homework from Larson, Hostetler, Edwards Calculus [15].

Large numbers of interiorization and coordination initiates.

Small numbers of generalization and reversal initiates.

No encapsulation initiates.

Interiorization

Students completed several procedures in completing the homework.

Coordination

- The limit exists and the function value exists but they are not equal. The graph is not continuous and there is a hole.
- The limit at $f(5)$ is $1/10$. Substitution doesn't work initially. Once the hole is "removed" we can find the limit of the function based on the all-but-one-point rule.
- The value of the function at c doesn't match the limit of the graph at c so the function is not continuous.

Generalization

$$\lim_{x \rightarrow 0} \sqrt{x}$$

- Some students said it was false because it was impossible to approach 0 from the left.
- Other students said it was true because as x gets closer and closer to 0, $f(x)$ gets closer and closer to 0.

Encapsulation

- No examples in the text book homework.

Reversal

- Construct counterexamples
- Explain why certain statements are false.

Examples of Peer Initiates in the Experimental Curriculum

Interiorization

–Several examples of students asking questions to clarify procedures.

Coordination

–Many examples of students coordinating ideas of the limit

What should I write for Carla?

She looks at all three strategies

If you get the same answer for each, you have three ways to verify that your answer is correct.

Generalization

–Very few examples of students generalizing concepts

So the same thing would have been applicable to what we just did?

Yeah.

Encapsulation

–No examples identified in the student work sessions

Reversal

–No examples identified in the student work sessions

Summary of Peer Initiates

- Large numbers of interiorization and coordination initiates.
- Small numbers of generalization initiates.
- No encapsulation or reversal initiates

Examples of Instructor Initiates in the Experimental Curriculum

Interiorization

- Lecture on finding limits algebraically.
- Students were asked to describe processes used to find limits

Coordination

- If a function is heading to positive infinity does the limit exist?
- How is the notion of removable discontinuity related to the division by zero issue?
- How is this problem similar or different from the previous problem?
- In light of what you now know, can you go back and solve the previous problem using a different strategy?

Encapsulation

- After looking at the formal definition of limit what does it mean to you?
- What does it really mean for a function to be continuous?

Reversal

- If a function is not continuous can a limit exist?
- If a limit does not exist what would the table look like?

Summary of Instructor Initiates

- Large numbers of interiorization and coordination initiates.
- Small numbers of generalization, encapsulation and reversal initiates

Examples of Curriculum Initiates in the Experimental Curriculum

- Role playing assists students to use multiple strategies

- Tom uses tables to find limits.
- George uses graphs to find limits.
- Alice uses algebraic simplification to find limits.
- Carla connects all the ideas to find limits.

Interiorization

- Many examples asking students to perform procedures.
- Carla asks George to explain how the graph shows the function approaching the same value as x approaches 2 from the left and the right. How will George answer Carla's question?
- Alice likes algebraic simplification. She claims in this case it is appropriate to plug 2 into $g(x)$ in order to determine the behavior of the function as x approaches 2 from the left and from the right. How would she answer the question? Demonstrate the strategy.

Coordination

- Many examples asking students to coordinate ideas.
- Will a function always approach the same number from both the left and the right? Write a paragraph. Include examples and counterexamples in your discussion. Discuss how this idea is related to other ideas in the unit.

–Question: Tom asks whether or not the function is continuous at $x = -3$. George likes graphs. How would George answer Tom's question?

Answer: The graph stops and continues at another place.

Question: Alice likes definitions. How would Alice answer Tom's questions?

Answer: The function is not continuous at $x = -3$ because plugging the value into the equation does not give a y -value.

Generalization

- A few questions in the experimental curriculum asked students to generalize ideas.
- Construct a graph with specific characteristics
- Explain whether or not it is possible for a limit to exist if a “zero in the denominator” results after plugging in the appropriate value.

Encapsulation

- There was one particular question that asked students to encapsulate the concept of limit.
- Carla decides to write a summary of this collection of limit lessons in her notebook. She wants to write a definition in her own words and she wants to include relevant examples and counterexamples in her notes. Help Carla complete her task.

Reversal

- There were several questions that asked students to reverse definitions or construct counterexamples.
- Construct a graph to show when a limit does not exist.
- Will a function always approach the same number from both the left and the right? Write a paragraph. Include examples and counterexamples in your discussion. Discuss how this idea is related to other ideas in the unit.

Summary of Curriculum Initiates

- Large numbers of interiorization, coordination and reversal initiates.
- A relatively small number of generalization initiates.
- One major initiate of encapsulation.

How Did the Control Curriculum Promote Reflective Abstraction?

- Instructor Initiates
 - Several interiorization and coordination examples. Few generalization or reversal examples.
 - Metaphors used to promote encapsulation.
- Peer Initiates
 - No collaborative group work sessions so there were no opportunities to identify peer initiates.
- Curriculum Initiates
 - Not designed to promote reflective abstraction.
- Individual Initiates
 - Few students were reflective on the text homework.

Interiorization

- He presented lectures focused on helping students learn algebraic procedures for finding limits.

Coordination

- He lectured by asking questions. Many of these questions helped students coordinate ideas.
- Is it possible to make this function seamless? Is it possible to define it so well so that this is continuous?

Teacher: Now we can use our substitution rule . . .
What can you say about the first condition that has to be met?

Student: It has to exist.

Teacher: It does not exist, right?

Student: Check the next one.

Teacher: I need all three to work and the first fails so is the function continuous at 2?

Student: No

Teacher. I don't need to check the others since I need all three to be true. So therefore the function is not continuous at x equals 2. Is it defined at 2? What is it? Is it removable? Can we define it? Do we have a choice?

Generalization/Reversal

Not a significant number of generalization or reversal examples.

Encapsulation

- Hook Metaphor to describe the process of finding a limit.
- First and last name metaphor to help students understand piecewise functions.

Summary of Instructor Initiates

- Lectures promoted interiorization.
- Questions promoted coordination.
- Metaphors were designed to promote encapsulation.
- Generalization and reversal occurred rarely.

Primary Conclusions

1. The students in the experimental section outperformed the students in the control section on a test of limits.
2. The students who improved from the pretest to the posttest engaged in reflective abstraction more often than the students who did not improve.
3. The experimental section was successful in promoting reflective abstraction through individual, peer, instructor and curriculum initiates.

Secondary Conclusions

1. The control section was successful in promoting reflective abstraction through instructor initiates.
2. Six students in the experimental section outperformed six students in the control section on a measure of explanation.
3. Encapsulation was initiated rarely and occurred rarely.
4. Instructor and peer initiates were most often interiorization and coordination.

5. Coordination was the most prevalent construct in the interview analysis.
6. The students who improved from pretest to posttest had three times as many generalization examples than the other students.
7. Students in the experimental section were demonstrated more individual initiates of reflective abstraction on the text homework than the students in the control section.

Limitations

1. Small sample sizes
2. Limited time frame.
3. Lack of randomization in assignment to the two sections.
4. Teacher effect (However, if a teacher effect were present, it may have favored the students in the control curriculum.)

Reflection

Can a curriculum that initiates reflective abstraction improve student performance on the concept of limit?

- The experimental curriculum was successful in promoting reflective abstraction through individual, peer, curricular, and instructor initiates.
- The control curriculum was not designed to promote reflective abstraction. However, as one might expect from a good instructor, the control curriculum promoted reflective abstraction through instructor initiates.
- The students in the experimental section outperformed the students in the control section on a test of the concept of limit.
- Both sections examined similar examples in class and completed the same homework exercises.
- For these reasons it is fair to conclude that the curriculum was a significant reason for the success of the students in the experimental section.
- An interesting unsuspected result was that students in the experimental section were better at written communication of mathematics than were the students in the control section. This indicates that opportunities to reflect on learning, together with regular writing assignments, may improve a student's written communication skills in mathematics.

- Analysis of the data, from the comparison subgroup, suggests that students with the greatest improvement engage in reflective abstraction more often than students with less improvement. Therefore one may argue that reflective abstraction is significant factor in student performance on the concept of limit. Further examination of this data shows that generalization may be the key to developing understanding of the concept of limit.

- This study demonstrates that a calculus curriculum can promote reflective abstraction. Furthermore, such a curriculum together with instructor, peer, curriculum and individual initiates, improves student performance and written communication on the concept of limit.

- Reflective abstraction is an effective tool for improving a student's performance in mathematics. The constructs of interiorization, coordination, encapsulation, generalization and reversal should be examined in the process of mathematics curriculum development. Teachers should promote reflective abstraction through instructor, peer, and curricular initiates. They should design problem sets that enable students to initiate reflective abstraction independently.

- The challenges of teaching and learning mathematics are substantial. Promoting reflective abstraction will enable teachers to help students meet this challenge.

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