What is Hindu Mathematics Anyway

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Hindu Mathematics (alias Vedic Mathematics) is based on 16 mathematical formulas known as sutras and claimed to solve some of the problems in Arithmetic, Algebra, Geometry, Analytic Geometry and Calculus in shorter time.

"Whatever is consistent with right reasoning should be accepted even though it comes from a child and whatever is inconsistent therewith ought to be rejected, although emanating from an elderly person."

Jagadguru Sankaracarya Sri Bharati Krsna Tirtha Maharaja
– Author of Vedic Mathematics –

Motilal Banarsidass Publishers Pvt. Ltd., Delh

APPLICATIONS IN ARITHMETIC

Application 1.

Conversion of a (VULGAR) fraction into a decimal.

Consider the rational number $\frac{1}{19}$

Note 1: Length of the cycle of repeating decimals

= divisor -1 = 19 - 1 = 18

Note 2: Product of the last digit of the denominator and the last digit of the recurring decimal must be 9. Thus the last digit of the recurring decimal is 1.

So $\frac{1}{19} = 0.\underbrace{\_\ \_\ \_\ \_\ \_\ \_\ \_\ \_\ \_\ \_\ \_\ \_\ \_\ \_\ \_\ \_\ \_\ 1}_{\text{Cycle of length 18}}$

EKADIKA-PURVA SUTRA

"By one more than the previous one"

Meaning: Last digit of the divisor is 9 and "the previous one" or the penultimate digit being 1 and "one more than the previous one" results in 1+1 = 2 and "By" means multiplication.

Methodology: Start with 1 as the last digit of the recurring decimal and proceed leftward multiplying by 2, each surplus digit is carried over to the left.

x 2 ← ↓ Start

Thus $\frac{1}{19}$ = . _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ 1

18 (over the whole cycle)

$\frac{1}{19}$ = . 0 5 2 6 3 1 5 7 8 ⋮ 9 4 7 3 6 8 4 2 1

carried over: 1 (under 5), 1 1 1 1 (under 1 5 7 8), 1 (under 4), 1 1 (under 3 6) ← carried over

Note: Divide the cycle into two halves (each of length 9)

0	5	2	6	3	1	5	7	8
9	4	7	3	6	8	4	2	1
9	9	9	9	9	9	9	9	9

Thus when the second half of the cycle is computed, the first half could be determined by 9th complementing.

Try: $\frac{1}{29}$, $\frac{1}{59}$
↓ ↓

9th complement ← ⋮ ← x 3

$\frac{1}{29}$ = . 0 3 4 4 8 2 7 5 8 6 2 0 6 8 ⋮ 9 6 5 5 1 7 2 4 1 3 7 9 3 1

carried over: 1 1 (under 6 5), 2 (under 5), 1 (under 7), 1 2 2 (under 4 1 3)

Application 2. (Multiplication)

Complement

Example 1 87 x 91 (base 100)

```
      87 - 13
         X
      91 - 09
     87-09  :
or   91-13  : + 17
              1
      78

7800 + 17 = (7917)
```

Example 2 38 x 44 (base = 100/2)

```
 38 : -12
 44 : -06
 32 : +72
 2

1600 + 72 = (1672)
```

Example 3 18 x 19 (base = 2x10)

```
   1 8 : -2
   1 9 : -1
2 x 17 :  2

340 + 2 = (342)
```

Example 4 108 x 97 (base 100)

```
108 :  + 08
 97 :  - 03
105 :  - 24
105  00 - 24    = (10476)
```

APPLICATIONS IN ALGEBRA

Application 3. (Division of Polynomials)

Horner's synthetic division extended for higher order divisors

Example 1

$$\frac{6x^4 + 13x^3 + 39x^2 + 37x + 45}{x^2 - 2x - 9}$$

```
2+9)  6    13    39    37    45
                  +
                 12    54

                       50    225

                             286   1287  (STOP)
      ________________________________________
           6     25    143   548   1332
```

$Q = 6x^2 + 25x + 143$ $R = 548x + 1332$

Eg 5 (Suppose leading coefficient of the divisor is not unit)

$$\frac{2x^5 - 9x^4 + 5x^3 + 16x^2 - 16x + 36}{2x^2 - 3x + 1}$$

Divisor = $2\left(x^2 - \frac{3}{2}x + \frac{1}{2}\right)$

$\frac{3}{2}$ $-\frac{1}{2}$) 2 -9 5 16 -16 36

$\frac{3}{2}$ -1

$-\frac{9}{2}$ 3

$-\frac{15}{2}$ $+ \frac{5}{2}$

$\frac{69}{4}$ $-\frac{23}{4}$ (STOP)

$Q = x^3 - 3x^2 - \frac{5x}{2} + \frac{23}{4}$

2 -6 -5 $\frac{23}{2}$ $\frac{15}{4}$ $\frac{121}{4}$

$R = \frac{15}{4}x + \frac{121}{4}$ $\frac{1}{2}\left(2 \;\; -6 \;\; -5 \;\; \frac{23}{2}\right)$ $\frac{15}{4}$ $\frac{121}{4}$

APPLICATION 4 (FACTORIZATION)

ADYAMADENA SUTRA – ALTERNATE ELIMINATION & RETENTION

Eg 1 Factorize $E = 2x^2 + 6y^2 + 3z^2 + 7xy + 11yz + 7zx$

Set z = 0 and factorize $2x^2 + 7xy + 6y^2 = (2x + 3y)(x + 2y)$ ____ (1)

Similarly setting y = 0 $2x^2 + 7xz + 3z^2 = (2x + z)(x + 3z)$ ____ (2)

☐ Filling in the gaps which we had created by leaving out z and y.

$E = (2x + 3y + z)(x + 2y + 3z)$

[Note: Setting $x = 0$ Ψ $6y^2 + 11yz + 3z^2 = (3y + z)(2y + 3z)$ confirms the above.]

Eg 2 $E = 3x^2 + 7xy + 2y^2 + 11xz + 7yz + 6z^2 + 14x + 8y + 14z + 8$

By eliminating 2 letters at a time

Set $y = z = 0$ Ψ $3x^2 + 14x + 8 = (x + 4)(3x + 2)$

Set $x = z = 0$ Ψ $2y^2 + 8y + 8 = (2y + 4)(y + 2)$

Set $x = y = 0$ Ψ $6z^2 + 14z + 8 = (3z + 4)(2z + 2)$

□ $E + (x + 2y + 3z + 4)(3x + y + 2z + 2)$

APPLICATION 5

Highest Common Factor

Lopana – Sthapana Sutra, Sankalana – Vyavakala process and the Adyamadya rule.

Alternate destruction of the highest and the lowest power by a suitable multiplication of the coefficient term and the addition/subtraction of the multiples.

Eg 1 HCF of $4x^3 + 13x^2 + 19x + 4$ and $2x^3 + 5x^2 + 5x - 4$

Methodology:

destroy the highest power

$$4x^3 + 13x^2 + 19x + 4$$
$$σ \quad \underline{4x^3 + 10x^2 + 10x - 8}$$
$$3x^2 + 9x + 12$$
$$= 3(x^2 + 3x + 4)$$

destroy the lowest power

$$4x^3 + 13x^2 + 19x + 4$$
$$ρ \quad \underline{2x^3 + 5x^2 + 5x - 4}$$
$$6x^3 + 18x^2 + 24x$$
$$= 6x(x^2 + 3x + 4)$$

Thus HCF $= x^2 + 3x + 4$

Check:

$4x^3 + 13x^2 + 19x + 4 = (x^2 + 3x + 4)(4x + 1)$
$2x^3 + 5x^2 + 5x - 4 = (x^2 + 3x + 4)(2x - 1)$

HCF = $x^2 + 3x + 4$

Eg 2 HCF of $4x^4 + 11x^3 + 27x^2 + 17x + 5$ and $3x^4 + 7x^3 + 18x^2 + 7x + 5$

destroy highest power

$$\begin{array}{rr} & 12x^4 + 33x^3 + 81x^2 + 51x + 15 \\ \sigma & \underline{12x^4 + 28x^3 + 72x^2 + 28x + 20} \\ & 5x^3 + 9x^2 + 23x - 5 \\ \sigma & \underline{5x^3 + 20x^2 + 45x + 50} \\ & -11x^2 - 22x - 55 \\ = & -11\,(\underline{x^2 + 2x + 5}) \end{array}$$

destroy lowest power

$$\begin{array}{rl} & 4x^4 + 11x^3 + 27x^2 + 17x + 5 \\ & \underline{3x^4 + 7x^3 + 18x^2 + 7x + 5} \quad \sigma \\ & x^4 + 4x^3 + 9x^2 + 10x \\ = & x\,(x^3 + 4x^2 + 9x + 10) \\ & \rho \\ & \underline{10x^3 + 18x^2 + 46x - 10} \\ & 11x^3 + 22x^2 + 55x \\ = & 11x\,(\underline{x^2 + 2x + 5}) \end{array}$$

□ HCF = $x^2 + 2x + 5$

Check:

$$4x^4 + 11x^3 + 27x^2 + 17x + 5 = (x^2 + 2x + 5)(4x^2 + 3x + 1)$$
$$3x^4 + 7x^3 + 18x^2 + 7x + 5 = (x^2 + 2x + 5)(3x^2 + x + 1)$$

HCF = $x^2 + 2x + 5$

APPLICATION 6

SOLVING OF EQUATIONS

SUNYAM SAMYASAMUCCAYA

When the Samuccaya is the same, that Samuccaya is zero.

The Samuccaya has so many meanings:

Eg 1

Consider solve: $\dfrac{3x + 4}{6x + 7} = \dfrac{x + 1}{2x + 3}$

This is of the form $\dfrac{N_1}{D_1} = \dfrac{N_2}{D_2}$

With $N_1 + N_2 = K(D_1 + D_2)$

Solution: Set $N_1 + N_2$ or $D_1 + D_2$ to zero.

Thus $4x + 5 = 0$ or $x = -5/4$.

Note: If the problem reduces to a quadratic form instead of linear, then the other (or 2nd) solution is obtained by setting the difference between the numerator and the denominator (of either side) to zero.

Eg 2

Solve: $$\frac{3x + 4}{6x + 7} = \frac{5x + 6}{2x + 3}$$

Since $N_1 + N_2 = D_1 + D_2$ one solution is obtained by setting $N_1 + N_2$ or $D_1 + D_2 = 0$ i.e. $8x + 10 = 0$ or $x = -5/4$.

Since the problem reduces to a quadratic form the other solution is obtained by setting $N_1 \sim D_1$ or $N_2 \sim D_2$ to zero.

i.e. $3x + 3 = 0$ or $x = -1$.

thus the solutions are $x = \frac{-5}{4}, -1$.

Eg 3

Solve: $$\frac{1}{x - 7} + \frac{1}{x + 9} = \frac{1}{x + 11} + \frac{1}{x - 9}$$

Here $D_1 + D_2 = D_3 + D_4$

By Sunyam Samuccaya, $D_1 + D_2 = D_3 + D_4 = 0$ yields the solution to this reduced linear equation.

Thus the solution is given by $2x + 2 = 0$ $x = -1$

Alt. LS $\frac{2x + 2}{(x - 7)(x + 9)}$ RS $\frac{2x + 2}{(x + 11)(x - 9)}$ then $x = -1$ is a solution

$x^2 + 2x - 63 = x^2 + 2x - 99$ Inconsistent/No other id.

Eg 4

$$\frac{1}{x-8} - \frac{1}{x-9} = \frac{1}{x-12} - \frac{1}{x-9}$$

Here $D_1 + D_2$ □ $D_3 + D_4$

but $\frac{1}{x-8} + \frac{1}{x-9} = \frac{1}{x-12} + \frac{1}{x-5}$ is of the form where

$$(x-8)(x-9) = (x-12)(x-5)$$

$$72 \neq 60 \qquad \text{No other sol.}$$

$$D_1 + D_2 = D_3 + D_4 = 2x - 7$$

By S.S. $2x - 7 = 0$ $x = \frac{7}{2}$

Eg 5

$$\frac{3}{3x+1} - \frac{6}{6x+1} = \frac{3}{3x+2} - \frac{2}{2x+1}$$

This does not satisfy $D_1 + D_2 = D_3 + D_4$

But using a common numerator viz 6 modifies the problem too

$$\frac{6}{6x+2} - \frac{6}{6x+1} = \frac{6}{6x+4} - \frac{6}{6x+3}$$

and rearranging the terms, we have

$$\frac{6}{6x+2} + \frac{6}{6x+3} = \frac{6}{6x+4} + \frac{6}{6x+1}$$

$$(6x+2)(6x+3) = (6x+4)(6x+1)$$

$$6 \neq 4 \qquad \text{No other sol.}$$

Which satisfies $D_1 + D_2 = D_3 + D_4 = 12x + 5$

and by S.S. the solution is given by $12x + 5 = 0$ $x = -5/12$.

Some interesting results

A. If $(x - 2a)^3 + (x - 2b)^3 = 2(x - a - b)^3$

then $x = a + b$

eg **SOLVE**: $(x - 149)^3 + (x - 51)^3 = 2(x - 100)^3$

$x = 100$

B. If $(x + a + b - c)^3 + (x + b + c - a)^3 = 2(x + b)^3$

then $x = -b$

eg **SOLVE**: $(x - 249)^3 + (x + 247)^3 = 2(x - 1)^3$

Noting $a + b - c = -249$

and $-a + b + c = 247$

adding $2b = -2$ or $b = -1$

Thus $x = 1$

C. $\left(\dfrac{x + a + d}{x + a + 2d}\right)^3 = \dfrac{x + a}{x + a + 3d}$ Where N_2, N_1, D_1, D_2 form an A.S.

The solution is then given by $N_1 + D_1 + N_2 + D_2 = 0$

i.e. $4x + 4a + 6d = 0$

or $x = \dfrac{-1}{2}(2a + 3d)$

Eg $\left(\dfrac{x - 5}{x - 7}\right)^3 = \dfrac{x - 3}{x - 9}$ which satisfies above condition and thus by S.S $4x - 24 = 0$ $x = 6$

(SOLVING CONTINUED)

MERGER TYPE:

Conditions: i) Any no. of terms on LS = single term on RS.
ii) Sum of ratio of numerator to X coefficient of denominator on LS = that of RS.

Eg 1. $\dfrac{3}{x + 1} + \dfrac{4}{x + 2} = \dfrac{7}{x + 3}$

which satisfies $\frac{3}{1} + \frac{4}{1} = \frac{7}{1}$

and the solution of this reduced linear equation is given by

$$\frac{3\,(1 - 3)}{x + 1} + \frac{4\,(2 - 3)}{x + 2} = 0$$

$$4x + 4 = -6x - 12 \quad \text{or} \quad x = -8/5$$

Eg 2. $$\frac{4}{2x - 1} + \frac{9}{3x - 1} = \frac{25}{5x - 1}$$

Although condition (ii) is satisfied i.e. $\frac{4}{2} + \frac{9}{3} = \frac{25}{5}$

Set the coefficients of x, the same

L.C.M of the coefficients 2, 3, 5 = 30

$$\frac{60}{30x - 15} + \frac{90}{30x - 10} = \frac{150}{30x - 6}$$

and the solution is given by $\frac{60\,(-15 + 6)}{30x - 15} + \frac{90\,(-10 + 6)}{30x - 10} = 0$

$$3x - 1 + 2x - 1 = 0 \qquad x = 2/5$$

Eg 3. $$\frac{3}{2x + 3} + \frac{2}{3x + 2} = \frac{2}{4x + 1}$$

Although $\frac{3}{2} + \frac{2}{3}$ □ $\frac{2}{4}$, above could be expressed as

$$\frac{2x + 3 - 2x}{2x + 3} + \frac{3x + 2 - 3x}{3x + 2} = \frac{2\,(4x + 1) - 8x}{4x + 1}$$

$$1 - \frac{2x}{2x + 3} + 1 - \frac{3x}{3x + 2} = 2 - \frac{8x}{4x + 1}$$

or $$\frac{2x}{2x + 3} + \frac{3x}{3x + 2} = \frac{8x}{4x + 1}$$

$x = 0$ or $$\frac{2}{2x + 3} + \frac{3}{3x + 2} = \frac{8x}{4x + 1}$$

here $\frac{2}{2} + \frac{3}{3} = \frac{8}{4}$

Now by setting the coefficients of x the same (L.C.M of 2, 3, 4 = 12)

$$\frac{12}{12x + 18} + \frac{12}{12x + 8} = \frac{24x}{12x + 3}$$

and the solution is given by $\frac{12\,(18 - 3)}{12x + 18} + \frac{12\,(8 - 3)}{12x + 8} = 0$

$$2\,(3x + 2) + 2x + 3 = 0$$

$$x = -7/8$$

Thus the solutions are x = 0, -7/8.

OTHER INTERESTING RESULTS:

TYPE 1: If $\frac{p}{(x + a)\,(x + b)} + \frac{q}{(x + b)\,(x + c)} + \frac{r}{(x + c)\,(x + a)} = 0$

then $x = -\,(pc + qa + rb) \;/\; (p + q + r)$

$$x = \frac{\text{sum of each numerator multiplied by the absent no. in the denominator.}}{\text{sum of the numerators}}$$

Eg. 1

SOLVE:

$$\frac{1}{6x^2 + 5x + 1} + \frac{1}{12x^2 + 7x + 1} + \frac{1}{8x^2 + 6x + 1} = 0$$

$$\frac{1}{(2x+1)\,(3x+1)} + \frac{1}{(3x+1)\,(4x+1)} + \frac{1}{(4x+1)\,(2x+1)} = 0$$

$$\frac{1/6}{(x + \frac{1}{2})\,(x + \frac{1}{3})} + \frac{1/12}{(x + \frac{1}{3})\,(x + \frac{1}{4})} + \frac{1/8}{(x + \frac{1}{4})\,(x + \frac{1}{2})} = 0$$

$$\square\; x = -\left[\frac{1}{6}\cdot\frac{1}{4} + \frac{1}{12}\cdot\frac{1}{2} + \frac{1}{8}\cdot\frac{1}{3}\right] \Big/ \left[\frac{1}{6} + \frac{1}{12} + \frac{1}{8}\right]$$

$x = -1/3$

Note: When p = q = r = 1 then $x = \frac{-1}{3}(a + b + c)$

Eg. 2

SOLVE:

$$\frac{x - 6}{(x - 2)(x - 3)} + \frac{x - 8}{(x - 3)(x - 4)} + \frac{x - 7}{(x - 4)(x - 2)} = \frac{3}{x + 1}$$

$$\frac{x - 6}{(x - 2)(x - 3)} - \frac{1}{x + 1} + \frac{x - 8}{(x - 3)(x - 4)} - \frac{1}{x + 1} + \frac{x - 7}{(x - 4)(x - 2)} - \frac{1}{x + 1} = 0$$

$$\frac{(x - 6)(x + 1)}{(x - 2)(x - 3)} - 1 + \frac{(x - 8)(x + 1)}{(x - 3)(x - 4)} - 1 + \frac{(x - 7)(x + 1)}{(x - 4)(x - 2)} - 1 = 0$$

$$\frac{-12}{(x - 2)(x - 3)} - \frac{20}{(x - 3)(x - 4)} - \frac{15}{(x - 4)(x - 2)} = 0$$

or $$\frac{12}{(x - 2)(x - 3)} + \frac{20}{(x - 3)(x - 4)} + \frac{15}{(x - 4)(x - 2)} = 0$$

and the solution is $x = -\frac{12(-4) + 20(-2) + 15(-3)}{12 + 20 + 15}$

$$x = \frac{133}{47}$$

TYPE 2: If $\frac{1}{AB} + \frac{1}{AC} = \frac{1}{AD} + \frac{1}{BC}$ where A, B, C, D form an A.P then Sumaccaye says "the ultimate and twice the penultimate"

One solution to this is given by D + 2C = 0

The other solution is obtained by setting the common difference to zero

SOLVE:

Eq. 1 $$\frac{1}{6x^2 + 7x + 2} + \frac{1}{8x^2 + 10x + 3} = \frac{1}{10x^2 + 13x + 4} + \frac{1}{12x^2 + 17x + 6}$$

$$\frac{1}{(2x + 1)(3x + 2)} + \frac{1}{(2x + 1)(4x + 3)} = \frac{1}{(2x + 1)(5x + 4)} + \frac{1}{(3x + 2)(4x + 3)}$$

here A = 2x + 1, B = 3x + 2, C = 4x + 3, D = 5x + 4 forming an A.P with common difference of x + 1

Thus by S.S solution is given by D + 2C = 5x + 4 + 2 (4x + 3)

$$\text{i.e. } 13x + 10 = 0 \text{ or } x = \frac{-10}{13}$$

The other solution is given by setting the common difference

$$x + 1 = 0 \quad \text{or} \quad x = -1$$

TYPE 3:

If $\frac{CA + D}{CB + E} = \frac{A}{B}$ then the S.S says "only the last terms"

i.e. $\frac{A}{B} = \frac{D}{E}$

Eq. 1 SOLVE:

$$\frac{(x + 1)(x + 6)}{(x + 3)(x + 5)} = \frac{x + 7}{x + 8} \quad \text{OR} \quad \frac{x^2 + 7x + 6}{x^2 + 8x + 15} = \frac{x + 7}{x + 8} \quad \text{OR} \quad \frac{x(x + 7) + 6}{x(x + 8) + 15} = \frac{x + 7}{x + 8}$$

Thus by S.S. $\frac{x + 7}{x + 8} = \frac{6}{15}$

$$15x + 105 - 6x - 48 = 0$$
$$9x = -57$$
$$x = \frac{-19}{3}$$

Eq. 2

$$(x + 2)(x + 3)(x + 11) = (x + 4)(x + 5)(x + 7)$$

Choose such that sum of the constants at the numerators are the same and also of the denominators.

$$\frac{(x + 2)(x + 3)}{(x + 4)(x + 7)} = \frac{x + 5}{x + 11}$$

$$\frac{x(x + 5) + 6}{x(x + 11) + 28} = \frac{x + 5}{x + 11}$$

which is of the form $\frac{CA + D}{CB + E} = \frac{A}{B}$

Thus the solution is $\frac{x+5}{x+11} = \frac{6}{28}$ Ψ $28x + 140 = 6x + 66$

$22x = -74$

$x = -37/11$

APPLICATION 7

(SOLVING CONTINUED)

QUADRATIC EQUATIONS & CALCULUS

Consider $ax^2 + bx + c = 0$ (a □ 0)

Solution is given by 1st differential $D_1 =$ ∀ $\sqrt{\text{Discriminant}}$

i.e. $D_1 = 2ax + b =$ ∀ $\sqrt{b^2 - 4ac}$

[or $x = \frac{1}{2a}$ [$-b$ ∀ $\sqrt{b^2 - 4ac}$] old friend (quadratic formula) in a new garb.

e.g. Consider $11x^2 + 7x + 7 = 0$

$D_1 = 22x + 7 =$ ∀ $\sqrt{7^2 - 4 \bullet 11 \bullet 7}$

$x = \frac{1}{22}$ [-7 ∀ $\sqrt{-259}$]

Note: When $a = 1$ i.e. $x^2 + bx + c = 0$

Then the first derivative w.r.t.x $D_1 = 2x + b$ = sum of the factors.

e.g. Consider $x^2 - 5x + 6 = 0$

factors are $(x - 2)(x - 3)$

1st differential $D_1 = 2x - 5 = (x - 2) + (x - 3)$

SOLVING CUBIC EQUATIONS

"By the completion"

Eq. 1 SOLVE: $x^3 - 6x^2 + 11x - 6 = 0$

$\frac{-6}{3} = -2$

or $x^3 - 6x^2 = -11x + 6$

But $(x - 2)^3 = x^3 - 6x^2 + 12x - 8$

$= -11x + 6 + 12x - 8$

$= x - 2$

Thus $x - 2 = 0$ or $x - 2 = \forall 1$

The solutions are $x = 2, 3, 1$

Eq. 2 SOLVE: $x^3 + 10x^2 + 27x + 18 = 0$

$\frac{10}{3} = 3+$

$= 4$ *(rounded up)*

$x^3 + 10x^2 = -27x - 18$

But $(x + 4)^3 = x^3 + 12x^2 + 48x + 64$

$= x^3 + 10x^2 + 2x^2 + 48x + 64$

$= -27x - 18 + 2x^2 + 48x + 64$

$= 2x^2 + 21x + 46$

$= (x + 4)(2x + 13) - 6$

Let $y = x + 4 \ \Psi \ y^3 = y(2y + 5) - 6$

$y^3 - 2y^2 - 5y + 6 = 0$

$(y - 1)(y^2 - y - 6) = 0$

$(y - 1)(y - 3)(y + 2) = 0$

Thus $y = -2, 1, 3$

or $x = y - 4 = -6, -3, -1$

CUBIC EQUATIONS AND CALCULUS

Consider $x^3 + bx^2 + cx + d = 0$ (with leading coefficient $a = 1$)

Observation 1: 1st derivative $D_1 = 3x^2 + 2bx + c$

= 3 product of factors taken two at a time

Observation 2: 2nd derivative $D_2 = 6x + b$

$= 2 ! \; 3$ factors

Example 1: $x^3 - 6x^2 + 11x - 6 = (x - 1)(x - 2)(x - 3)$

Observation 1: $D_1 = 3x^2 - 12x + 11$

$= (x - 1)(x - 2) + (x - 2)(x - 3) + (x - 3)(x - 1)$

$= x^2 - 3x + 2 + x^2 - 5x + 6 + x^2 - 4x + 3 = 3x^2 - 12x + 1$

Observation 2: $D_2 = 6x - 12$

$= 2 ! \{(x - 1) + (x - 2) + (x - 3)\}$

$= 2(3x - 6)$

SOLVING CONTINUED

QUARTIC/BIQUADRATIC EQUATIONS
(PURANAMA METHOD)

Example 1:

SOLVE: $x^4 - 20x^3 + 137x^2 - 382x + 360 = 0$

$x^4 - 20x^3 = -137x^2 + 382x - 360 = \frac{-20}{4} = -5$

But: $(a - b)^4 = a^4 - 4a^3 b + 6a^2 b^2 - 4ab^3 + b^4$

So $(x - 5)^4 = x^4 - 20x^3 + 150x^2 - 500x + 625$

$= -137x^2 + 382x - 360 + 150x^2 - 500x + 625$

$= 13x^2 - 118x + 265$

$= (x - 5)(13x - 53) = (x - 5)(13(x - 5) + 12)$

Let $x - 5 = y$ then $y^4 = y(13y + 12)$

or $y^4 - 13y^2 - 12y = 0$

$$y\,(y^3 - 13y - 12) = 0$$

$$y\,(y + 1)\,(y^2 - y - 12) = 0$$

$$y\,(y + 1)\,(y - 4)\,(y + 3) = 0$$

So $y = -3, -1, 0$ or 4

and $x = y + 5 = 2, 4, 5$ or 9

EXAMPLE 2

Solve: $(x + 7)^4 + (x + 5)^4 = 706$ ______ (1)

Let a be the average of the two binomials
$x + 5$ and $x + 7$
i.e. $a = x + 6$

So (1) becomes $(a + 1)^4 + (a - 1)^4 = 706$

or $(a^4 + 4a^3 + 6a^2 + 4a + 1) + (a^4 - 4a^3 + 6a^2 - 4a + 1) = 706$

$a^4 + 6a^2 - 352 = 0$

$(a^2 - 16)\,(a^2 + 22) = 0$

$a = \forall 4$ or $\forall \sqrt{22i}$

So $x = a - 6 = -10, -2, -6 \forall \sqrt{22i}$

QUARTIC/BIQUADRATIC EQUATIONS AND CALCULUS

Consider $x^4 + bx^3 + cx^2 + dx + e = 0$ (with leading coefficient a = 1)

Observation 1:

1st derivative $D_1 = 4x^3 + 3bx^2 + 2cx + d$

= 3 product of factors taken three at a time

Observation 2:

2nd derivative $D_2 = 12x^2 + 6bx + 2c$

= 2! 3 product of factors taken two at a time

Observation 3:

3rd derivative D_3 = 24x + 6b

= 3! 3 factors

Example 1

$x^4 - 20x^3 + 137x^2 - 382x + 360 = (x - 2)(x - 4)(x - 5)(x - 9)$

Observation 1: $D_1 = 4x^3 - 60x^2 + 274x - 382$

$= (x - 2)(x - 4)(x - 5) + (x - 2)(x - 4)(x - 9) + (x - 4)(x - 5)(x - 9) + (x - 2)(x - 5)(x - 9)$

Observation 2: $D_2 = 12x^2 - 120x + 274$

$= 2\{(x - 2)(x - 4) + (x - 2)(x - 5) + (x - 2)(x - 9) + (x - 4)(x - 5) + (x - 4)(x - 9) + (x - 5)(x - 9)\}$

Observation 3: $D_3 = 24x - 120 = 6\{(x - 2) + (x - 4) + (x - 5) + (x - 9)\}$

For a 5th order polynomial equation in one variable x with leading coefficient unit,

i.e. $x^5 + bx^4 + cx^3 + dx^2 + ex + f = 0$ (with a = 1)

1st derivative D_1 = 3 product of factors taken 4 at a time

2nd derivative D_2 = 2! 3 product of factors taken 3 at a time

3rd derivative D_3 = 3! 3 product of factors taken 2 at a time, and

4th derivative D_4 = 4! 3 factors

Example:

$x^5 + 15x^4 + 85x^3 + 225x^2 + 274x + 120 = (x + 1)(x + 2)(x + 3)(x + 4)(x + 5)$

Special Types of Quadratic Equations

Type 1 (deals with reciprocals)

Example 1:

Solve: $x + \frac{1}{x} = \frac{17}{4} = 4\frac{1}{4} = 4 + \frac{1}{4}$

$x = 4$ or $\frac{1}{4}$

Example 2:

Solve: $\frac{x+4}{x-4} + \frac{x-4}{x+4} = \frac{10}{3} = 3\frac{1}{3} = 3 + \frac{1}{3}$

Thus: $\frac{x+4}{x-4} = 3$ or $\frac{x+4}{x-4} = \frac{1}{3}$

$x = 8$ or -8

Example 3:

Solve: $\frac{2x+11}{2x-11} + \frac{2x-11}{2x+11} = \frac{193}{84}$ $(= 2\frac{25}{84}$ or $2 + \frac{25}{84}$ does not help

$= \frac{12}{7} + \frac{7}{12}$

Thus $\frac{2x+11}{2x-11} = \frac{12}{7}$ or $\frac{2x+11}{2x-11} = \frac{7}{12}$

$x = \frac{209}{10}$ or $\frac{-209}{10}$

Example 4:

Solve: $\frac{5x+9}{5x-9} - \frac{5x-9}{5x+9} = \frac{56}{45}$

$= \frac{9}{5} - \frac{5}{9}$

Thus $\frac{5x+9}{5x-9} = \frac{9}{5}$ or $\frac{5x+9}{5x-9} = \frac{-5}{9}$

$x = \frac{63}{10}$ or $\frac{-18}{35}$

Note: The denominator of the RS has to be factor into two factors so that the sum (or difference) of squares of these factors is the numerator of the RS.

Type 2

Methodology: Oneness of the sum of the numerators/denominators gives one solution and the oneness of the difference between the numerator and the denominator on either side gives the other solution.

Example 1

Solve: $$\frac{16x - 3}{7x + 7} = \frac{2x - 15}{11x - 25}$$

of the form $\frac{N_1}{D_1} = \frac{N_2}{D_2}$

Where $D_1 + D_2 = N_1 + N_2$ $\quad \frac{(18x - 18)}{(-9x - 10)} = \frac{1}{2}$

and $N_1 - D_1 = D_2 - N_2$

and one solution is given by

$$D_1 + D_2 = N_1 + N_2 = 0$$

ie $8x - 18 = 0$ or $x = 1$

And the other solution is given by

$$N_1 - D_1 = D_2 - N_2$$

$$9x - 10 = 0 \text{ or } x = \frac{10}{9}$$

Type 3

$$\frac{a}{1x + a} + \frac{b}{1x + b} = \frac{c}{1x + c} + \frac{d}{1x + d}$$

If the ratios of the coefficients $\frac{a}{1} + \frac{b}{1} = \frac{c}{1} + \frac{d}{1}$ (ie $a + b = c + d$)

and $\frac{a}{a} + \frac{b}{b} = \frac{c}{c} + \frac{d}{d}$ (=2)

then one solution is $x = 0$

and the other is given by $D_1 + D_2$ $\quad D_3 + D_4 = 0$

or simply $x = -\frac{1}{2}(a+b) = -\frac{1}{2}(c+d)$

EXAMPLE:

$$\frac{3}{1x+3} + \frac{4}{1x+4} = \frac{2}{1x+2} + \frac{5}{1x+5}$$

which satisfies above form

Thus $x = 0$ and $D_1 + D_2 = D_3 + D_4 = 2x + 7 = 0 \quad x = \frac{-7}{2}$

Try: $\frac{1}{x+1} + \frac{4}{x+4} = \frac{2}{x+2} + \frac{3}{x+3} \quad x = 0, x = \frac{-5}{2}$

Solve for x: $a(x+b)(x+c)(x+d) + b(x+a)(x+c)(x+d)$
$= c(x+a)(x+b)(x+d) + d(x+a)(x+b)(x+c)$ $\quad S + a + b = c + d$ and $ab \neq cd$

LS: $a(x+b)(x^2 + (c+d)x + cd) + b(x+a)(x^2 + (c+d)x + cd)$
$= ax^3 + a(b+d+d)x^2 + a(b(c+d) + cd)x + abcd$
$+ bx^3 + b(a+c+d)x^2 + b(a(c+d) + cd)x + abcd$
$= (2ab)x^2 + 2ab(c+d) + (a+b)cd\,x$

RS: $c(x+d)(x^2 + (a+b)x + ab) + d(x+e)(x^2 + (a+b)x + ab)$
$= cx^3 + c(d+a+b)x^2 + c(d(a+b) + ab)x + abcd$
$+ dx^3 + d(c+a+b)x^2 + d(c(a+b) + ab)x + abcd$
$(2cd)x^2 + (2cd(a+b) + ab(c+d))x$
$2abx^2 + (2ab(c+d) + cd)x = 2cd\,x^2 + (2cd(a+b) + ab)x$
$2abx^2 + (abc + abd)x = 2cd\,x^2 + acd + bcd)x$
$2(ab - cd)x^2 + (abc + abd - acd - bcd)\;x = 0$
$x = 0$ or $x = -\frac{abc + abd - acd - bcd}{2(ab - cd)} = -\frac{(a+b)(ab - bc)}{2(ab - cd)} = \frac{-(a+b)}{2}$
$= \frac{-(c+d)}{2}$

Application 8

SIMPLIFICATION

Type 1 Proof

Show: $\frac{1}{(x+a)(x+a+d)} + \frac{1}{(x+a+d)(x+a+2d)} + \frac{1}{(x+a+2d)(x+a+3d)} = \frac{3}{(x+a)(x+a+3d)}$

LS: $\dfrac{(x + a + 2d)(x + a + 3d) + (x + a)(x + a + 3d) + (x + a)(x + a + d)}{LCD}$

$= \dfrac{x^2 + (2a+5d)\,x + (a^2 + 5ad+6d^2) + x^2 + (2a+3d)\,x + (a^2 + 3ad) + x^2 + (2a+d)\,x + a^2 + ad}{LCD}$

$= \dfrac{3\,[x^2 + (2a + 3d)\,x + \overbrace{(a^2 + 3ad + 2d^2)}^{(a + cd)(a + 2d)}\,]}{(x + a)(x + a + d)(x + a + 2d)(x + a + 3d)} = \dfrac{3\,[x + (a + d)\,]\,[x + (a + 2d)\,]}{(x + a)(x + a + d)(x + a + 2d)(x + a + 3d)}$

$= \dfrac{3}{(x + a)(x + a + 3d)}$

Simplify: $\dfrac{1}{(x + a)\,(x + b)} + \dfrac{1}{(x + b)\,(x + c)} + \dfrac{1}{(x + c)\,(x + d)}$ where

x + a, x + b, x + c, x + d in an A.P

$= \dfrac{1 + 1 + 1}{(x + a)\,(x + d)}$ or $\dfrac{\text{sum of numerators}}{\text{product of first and last term of the A.P}}$

$= \dfrac{3}{(x + a)\,(x + d)}$

Example 1

Simplify: $\dfrac{1}{(x + a)\,(2x + 3a)} + \dfrac{1}{(2x + 3a)\,(3x + 5a)} + \dfrac{1}{(3x + 5a)\,(4x + 7a)}$

$= \dfrac{3}{(x + a)\,(4x + 7a)}$

In general

$\dfrac{1}{(x + a)\,(x + a + d)} + \dfrac{1}{(x + a + d)\,(x + a + 2d)} + \dfrac{1}{(x + a + (A - 1)\,d)\,(x + a + nd)}$

$= \dfrac{n}{(x + a)\,(x + a + nd)}$

Example 2

Simplify: $\dfrac{1}{x^2 - 3x + 2} + \dfrac{1}{x^2 - 5x + 6} + \dfrac{1}{x^2 - 7x + 12} + \dfrac{1}{x^2 - 9x + 20} + \dfrac{1}{x^2 - 11x + 30}$

$$= \frac{1}{(x-1)(x-2)} + \frac{1}{(x-2)(x-3)} + \frac{1}{(x-3)(x-4)} + \frac{1}{(x-4)(x-5)} + \frac{1}{(x-5)(x-6)}$$

$$= \frac{5}{(x-1)(x-6)}$$

TYPE II:

Simplify: $$\frac{a-b}{(px+a)(px+b)} + \frac{b-c}{(px+b)(px+c)} + \frac{c-d}{(px+c)(px+d)}$$

$$= \frac{a-d}{(px+a)(px+d)}$$

Show: $$\frac{a-b}{(px+a)(px+b)} + \frac{b-c}{(px+b)(px+c)} + \frac{c-d}{(px+c)(px+d)} = \frac{a-d}{(px+a)(px+d)}$$

$$\frac{(a-b)(px+c)(px+d) + (b-c)(px+a)(px+d) + (c-d)(px+a)(px+b)}{(px+a)(\rho x+b)(\rho x+c)(\rho x+d)}$$

$(a - b)\ [p^2x^2 + p(c + d)x + cd] + (b - c)[p^2x^2 + p(a + d)x + ad]\ [p^2x^2 + p(a + b)x + ad]$

LCD:

$p^2(a-d)x^2 + p[(a-b)(c+d) + (b-c)(a+d) + (c-d)(a+b)]x + (a-b)cd + (b-c)ad + (c-d)ab/LC)$

$$p^2(a-d)x^2 + p\left[\frac{ab - cd + ca - bd}{(a-d)(b+c)}\right]x + bc(a-d)$$

$(a - d)[p^2x^2 + (b + c)\ bx + bc]$ /LCD

$$\frac{(a-d)(px+b)(px+c)}{(bx+a)(bx+b)(px+c)(px+d)} = \frac{a-d}{(bx+a)(px+d)} \text{ and } = 0 \text{ when } a = d$$

Note: If a = d then the answer = 0

Example 1: $$\frac{-1}{(x+7)(x+8)} + \frac{-2}{(x+8)(x+10)} + \frac{-14}{(x+10)(x+24)} = \frac{13}{(x+7)(x+24)}$$

Example 2: $$\frac{1}{(x+7)(x+8)} + \frac{2}{(x+8)(x+10)} - \frac{3}{(x+10)(x+7)} = 0$$

Application 9

Partial Fractions

Type 1 Linear factors without repeated factors in reduced form.

PARAVARTYA SUTRA (simplified version of Heavyside method)

Example 1 Find Partial Fraction of $E = \dfrac{3x^2 + 12x + 11}{(x+1)(x+2)(x+3)}$ ______ (1)

Let $E = \dfrac{A}{x+a} + \dfrac{B}{x+2} + \dfrac{C}{x+3}$

Methodology: To solve for A, set the denominator of A to zero, thus x = -1 and substitute this in (1) without the factor x + 1.

Justification: $\left.\dfrac{3x^2 + 12x + 11}{(x+1)(x+2)(x+3)}\right| = \dfrac{A}{x+1} + \dfrac{B}{x+2} + \dfrac{C}{x+3}$

$$= \frac{A(x+2)(x+3) + B(x+1)(x+3) + C(x+1)(x+2)}{\text{LCD}}$$

To determine A, set x = -1 on both sides of the numerator.

III^y for B set x = -2 and for C set x = -3 on both sides of the numerator.

Thus $A = \left.\dfrac{3x^2 + 12x + 11}{(x+2)(x+3)}\right|_{x=-1} = \dfrac{3 - 12 + 11}{1 \bullet 2} = \dfrac{2}{2} = 1$

III^y $B = \left.\dfrac{3x^2 + 12x + 11}{(x+1)(x+3)}\right|_{x=-2} = \dfrac{12 - 24 + 11}{(-1)(1)} = \dfrac{-1}{-1} = 1$

and $C = \left.\dfrac{3x^2 + 12x + 11}{(x+1)(x+2)}\right|_{x=-3} = \dfrac{17 - 36 + 11}{(-2)(-1)} = \dfrac{2}{2} = 1$

Hence $E = \dfrac{3x^2 + 12x + 11}{(x+1)(x+2)(x+3)} = \dfrac{1}{x+1} + \dfrac{1}{x+2} + \dfrac{1}{x+3}$

Type 2 one linear factor, repeated.

Example: Find partial fractions of $E = \dfrac{x^3 + 3x + 1}{(1-x)^4}$

Traditional MTD: $\dfrac{x^3 + 3x + 1}{(1-x)^4} = \dfrac{A}{1-x} + \dfrac{B}{(1-x)^2} + \dfrac{C}{(1-x)^3} + \dfrac{D}{(1-x)^4}$

Methodology: Let $1 - x = p$ or $x = 1 - p$

Thus, $E = \dfrac{(1-p)^3 + 3(1-p) + 1}{p^4} = \dfrac{1 - 3p + 3p^2 - p^3 + 3 - 3p + 1}{p^4}$

$= \dfrac{5}{p^4} - \dfrac{6p}{p^4} + \dfrac{3p^2}{p^4} - \dfrac{p^3}{p^4}$

$= \dfrac{5}{(1-x)^4} - \dfrac{6}{(1-x)^3} + \dfrac{3}{(1-x)^2} - \dfrac{1}{1-x}$

PARTIAL FRACTIONS & CALCULATIONS

Type 3 Repeated factors linear and nonlinear, not necessarily one factor.

Example 1 Find Partial Fractions of

$$E = \frac{1}{x^3 - x^2 - x + 1} = \frac{1}{(x-1)^2(x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1}$$

$x^2(x-1) - (x-1)$
$(x-1)\ (x^2-1)$
$(x-1)^2\ (x+1)$

The numerator corresponding to the factor which has no repeats (i.e., C) and the numerator corresponding to the factor with the highest number of repetitions (i.e., B) is determined as before (Type 1).

$$C = \left.\frac{1}{(x-1)^2}\right|_{x=-1} = \frac{1}{4}$$

$$B = \left.\frac{1}{(x+1)}\right|_{x=1} = \frac{1}{2}$$

Thus $E = \dfrac{1}{(x-1)^2(x+1)} = \dfrac{A}{x-1} + \dfrac{½}{(x-1)^2} + \dfrac{¼}{x+1}$

or $1 = A(x^2 - 1) + ½(x+1) + ¼(x-1)^2$

Now differentiate, yes differentiate w • r • t • x :

$0 = 2Ax + ½ + ¼ \bullet 2(x-1)$

$= (2A + ½)x$

for all values of x, A = -¼

Thus $E = \dfrac{1}{(x-1)^2(x+1)} = \dfrac{-\frac{1}{4}}{x-1} + \dfrac{\frac{1}{2}}{(x-1)^2} + \dfrac{\frac{1}{4}}{x+1}$

Justification:

$$1 = A(x-1)(x+1) + B(x+1) + C(x-1)^2$$

To determine C: Set x = -1, $1 = C(-1-1)^2 \;\Psi\; C = \frac{1}{4}$

To determine B: Set x = 1, $1 = B(1+1) \;\Psi\; B = \frac{1}{2}$

TRADITIONAL MTD:

To determine A: Compare coeff: $x^2 \bullet 0 = A + \frac{1}{4}$ or $A = -\frac{1}{4}$

or compare constants: $1 = -A + \frac{1}{2} + \frac{1}{4}$
$A = -\frac{1}{4}$

EXAMPLE 2

Find partial fractions of $E = \dfrac{x^5 - x^4 - 3x + 5}{x^4 - 2x^3 + 2x^2 - 2x + 1}$

$$= x + 1 + \frac{-2x+4}{(x^2+1)(x-1)^2}$$

Let $\dfrac{-2x+4}{(x^2+1)(x-1)^2} = \dfrac{Ax+B}{x^2+1} + \dfrac{C}{x-1} + \dfrac{D}{(x-1)^2}$

As before $D = \left.\dfrac{-2x+4}{x^2+1}\right|_{x=1} = 1$

So $-2x + 4 = (Ax+B)(x-1)^2 + C(x-1)(x^2+1) + 1(x^2+1)$

differentiate w • r • t • x : $-2 = A(x-1)^2 + 2(Ax+B)(x-1) + C(3x^2 - 2x + 1) + 2x$ ______ (1)

$x = 1 \;\Psi\; -2 = C(2) + 2 \bullet 1$ or $C = -2$

Thus ①: $-2 = A(x-1)^2 + 2(Ax+B)(x-1) - 6x^2 + 4x - 2 + 2x$ ______ (2)

differentiate w • r • t • x : $0 = 2A(x-1) + 2A(x-1) + 2(Ax+B) - 12x + 6$

$$0 = 4A(x - 1) + 2(Ax + B) - 12x + 6$$

differentiate w • r • t • x : $0 = 4A + 2A - 12$ or $A = 2$

Thus ②: Ψ $-2 = 2(x - 1)^2 + 2(2x + B)(x - 1) - 6x^2 + 6x - 2$

for any value of x other than 1, say $x = 0$

$$-2 = 2 + 2 \bullet B(-1) - 2 \ \Psi \ B = 1$$

Hence $E = \dfrac{x^5 - x^4 - 3x + 5}{x^4 - 2x^3 + 2x^2 - 2x + 1} = x + 1 + \dfrac{2x + 1}{x^2 + 1} - \dfrac{2}{x - 1} + \dfrac{1}{(x - 1)^2}$

Application 10

Analytical Conies

Eg. ration of a straight line passing through two distinct points (x_1,y_1), (x_2,y_2) is given by:

$$(y_1 - y_2)\,x - (x_1 - x_2)\,y = x_2y_1 - x_1y_2$$

↗ difference in y coordinates; ↗ difference in x coordinates; ↗ inner product; ↖ outer product

Example: Find the equation of the straight line passing through (9,17) and (7,-2)

$$(17 - (-2)\ x - (9-7)\,y = 17\,(7) - (9\,(-2)\,)$$

$$19x - 2y = 119 + 18 = 137$$

Note: (i) If $x_2y_1 = x1y_2$ then the straight line will pass through the origin.
$ax + by = 0$

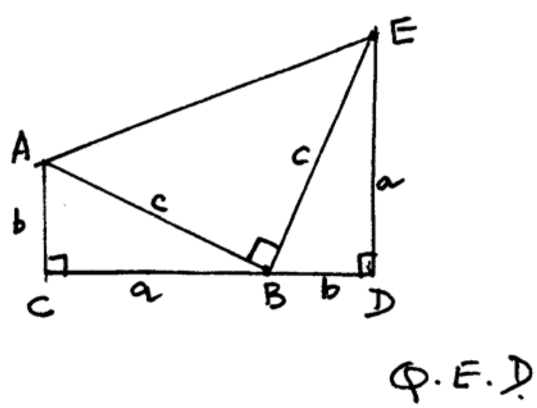
(ii) If $x_1 = x_2$ then it's a vertical line x = constant and
if $y_1 = y_2$ Ψ then it's a horizontal line y = constant.

Application 11

Geometry

EXAMPLE 1. PROOF OF PYTHOGORA'S THEOREM.

Proof 1. (without words?)

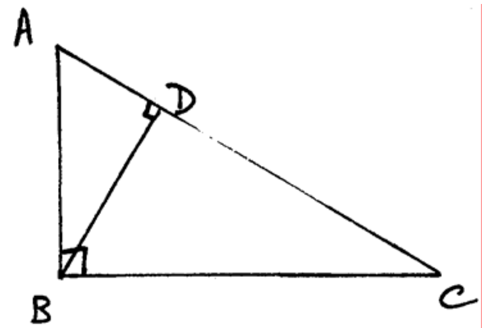


Area of trapezoid ACDE -

= areas of Δ^s ABC + ABE + BDE

$$\frac{1}{2}(a+b)^2 = \frac{1}{2}ab + \frac{1}{2}c^2 + \frac{1}{2}ab$$

$$a^2 + b^2 = c^2$$



Proof 2.

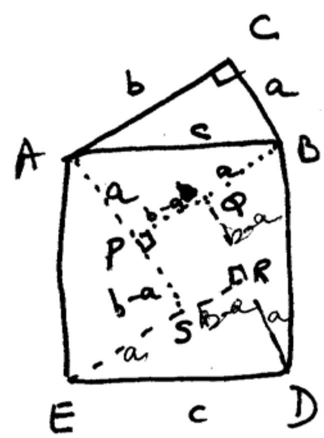
Δ^s ABC, BDC, ADB are similar.
But areas of similar Δ^s are proportional to the squares on the homologous (corresponding) sides.

Consider Δ^s BDC, ABC $\quad \frac{BC^2}{AC^2} = \frac{\Delta BDC}{\Delta ABC}$ ______ (1)

Δ^s BDC, ABC $\quad \frac{AB^2}{AC^2} = \frac{\Delta ADB}{\Delta ABC}$ ______ (2)

(1) + (2) $\quad \frac{AB^2 + BC^2}{AC^2} = 1$

or $\quad AB^2 + BC^2 = AC^2$



Proof 3.

Complete the rectangle ACBP
Extend AP to S s • t AS = b (or PS = b-a)
Choose Q on PB s • t PQ = b-a
Complete the square SPQR of length b-a.

It could be shown that E-S-R and ER2PB, AC and D-R-Q and DQ2SA, BC.

Now area of square ABDE = area of square PQRS + 4 area of ΔABP

Thus $\quad C^2 = (b - a)^2 + 4 \cdot \frac{1}{2} \cdot b \cdot a$

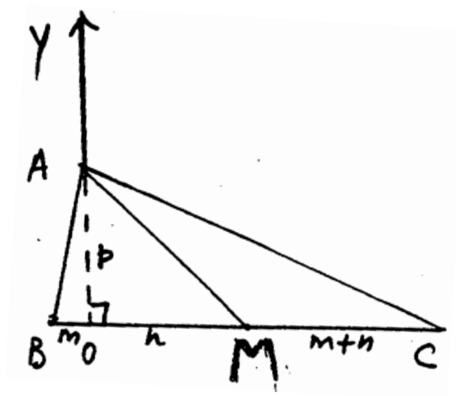
$$C^2 = a^2 + b^2$$

EXAMPLE 2.

APOLLONIUS THEOREM:

In any ΔABC, if M is the midpoint of BC (i.e., AD, median through A) then

$$AB^2 + AC^2 = 2\,(AM^2 + BM^2)$$



Proof (without words)

$AB^2 + AC^2 = m^2 + p^2 + (m+2n)^2 + p^2 = m^2 + p^2 + m^2 + 4mn + 4n^2 + p^2$

$= 2\,[\,(n^2 + p^2) + (m + n)^2\,] = 2m^2 + 4mn + 4n^2 + 2p^2$

$= 2\,[\,AM^2 + BM^2].$

Q.E.D.

-END-