An Investigation of Approaches and Strategies for Resolving Students' Misconceptions about Probability in Introductory College Statistics

Leonid Khazanov
Borough of Manhattan Community College,
Mathematics Department
199 Chambers Street, NY, NY 10007
Lkhazanov@bmcc.cuny.edu

Abstract: Research has documented that students frequently hold misconceptions about probability that are not necessarily resolved by traditional instruction. The purpose of this study was threefold: survey college professors of introductory applied statistics about their awareness of students' misconceptions, design and validate a test instrument to identify some prevalent misconceptions of probability, and investigate approaches and strategies used by college professors to facilitate the resolution of these misconceptions.
The results revealed that, in the opinion of survey respondents, misconceptions about probability interfere with students' ability to master inferential statistics. Some evidence was obtained that instructors who targeted misconceptions directly in instruction achieved better results in facilitating the resolution of misconceptions than those who used formal instruction.

Introduction: My interest in probability and misconceptions about probability may be traced back to the time when I was a student at the then Leningrad Pedagogical Institute in Russia. A roommate of mine who was a young mathematical prodigy from Siberia used to play a game called Sport Lotto, which is similar to our state lotteries. The aim of the game was to choose 6 numbers that would match the numbers drawn by the Lottery Officials. The prize was huge: 10,000 rubles. One could buy the fanciest car in Russia for that money.
I noticed that my friend never selected numbers in order, like 2, 3, 4, 5, 6 and 7. When I asked him why, he responded that numbers in order never win. He was sure that a combination that looked random had a better chance of winning. Even after we evaluated the probability of each combination my friend refused to choose numbers in order. On some emotional level the chances of winning with the combination of numbers that looked random seemed better.
Only many years later did I find out that about the same time my friend and I had our discussion, Daniel Kahneman and Amos Tversky (1973) published a ground breaking paper on misconceptions people have about probability. The misconception my friend entertained was described in that paper and was referred to as '**representativeness**.'
People who have this misconception estimate the likelihood of an event based on how well an outcome represents some aspect of the parent population. In the case of my friend, the argument could be as follows: the Lottery Officials draw the numbers at random; therefore the combination should look random. Other common misconceptions investigated in this study were the **equiprobability bias** (attributing the same probability in a random experiment to different events regardless of their actual chances; Lecoutre & Rezrazi,1998) and **outcome orientation**

(perceiving each trial as a separate individual phenomenon and treating the probability of an occurrence as a certainty rather than a measure of likelihood; Konold 1989, 1995).

A colleague of mine at BMCC Professor Chris McCarthy (personal communication, spring 2005) examined possible origins of these misconceptions.
By definition, if A is a subset of S, then relative to S, the probability of A is:

$$P(A) = \frac{\text{measure of A}}{\text{measure of S}}$$

In the finite case, the measure is often counting, which leads to

$$P(A) = \frac{\text{count}(A)}{\text{count}(S)}$$

This correct formulation of probability when over-applied may result in the *equiprobability misconceptions*.
For example, people often say,"either it will happen or it won't",
mathematically:

S = {it happens, it doesn't happen}

And hence, incorrectly,

$$P(\text{it happens}) = \frac{\text{count}(\{\text{it happens}\})}{\text{count}(S)} = \frac{1}{2}$$

In some sense, the equiprobability misconception is very self-centered. It assigns probability based on the number of categories **we** assign to a process, rather than based on the process itself.

Outcome orientation misconception has its roots in deterministic reasoning. There tends to be a lumping of probabilities into three basic states, essentially {0%, 50%, 100%} which roughly corresponds to {won't happen, no clue, will happen}.
*A probability of 95% seems to represent "it **will** happen, but let's play it safe, and say 95%".*

Purpose of study: The purpose of the study was threefold:

- Investigate college professors' perspectives on students' misconceptions of probability
- Design and validate an assessment instrument
- Investigate approaches and strategies used by college professors to facilitate students' conceptual understanding of probability and unlearn misconceptions

In regard to the first purpose, I wanted to know the extent to which college professors are aware of their students' misconceptions, whether they address misconceptions in teaching, and what approaches and strategies they believed to be productive in resolving the misconceptions. I also wanted to find out if instructors were interested in professional development designed to improve their knowledge on how to address misconceptions in instruction. To accomplish that aim I designed a questionnaire and had 66 statistics teachers from a number of colleges and universities answer it. The questionnaire probed instructors' opinions on a broad range of issues pertaining to students' misconceptions about probability. It is comprised of ten items. The majority of questions required that respondents made a choice from a proposed set of answers. At the same time, where appropriate, respondents were offered an opportunity to indicate an answer that was not listed in the answer choices. A design that included an open ended response option afforded an opportunity to gather the full spectrum of instructors' opinions. The format of questions varied from multiple-choice - to arranging items in order of importance - to rating items on a Likert-type scale. A typical set of items from the questionnaire is presented below, along with the rationale for including them in the survey.

Set 3. Please, evaluate the following statements:

		Strongly disagree	**Disagree**	**Neutral**	**Agree**	**Strongly Agree**
1	The study of probability is important for all college students	**1**	**2**	**3**	**4**	**5**
2	A large proportion of college students have misconceptions about probability	**1**	**2**	**3**	**4**	**5**
3	Misconceptions of probability adversely affect comprehension of statistics	**1**	**2**	**3**	**4**	**5**
4	Effective strategies exist for eliminating misconceptions of probability	**1**	**2**	**3**	**4**	**5**
5	Students misconceptions of probability can be significantly reduced during the time allocated to this topic in a typical introductory statistics course	**1**	**2**	**3**	**4**	**5**

Set 3 was aimed at probing instructors' opinions on some key issues pertaining to the study of probability in general and misconceptions in particular. My objective was to investigate whether instructors put a high premium on misconceptions and whether they believed it was worthwhile and possible to deal with them in the first statistics course.

Concerning the second purpose, I needed an instrument for the experimental phase of the study. I sought an instrument that would identify most common misconceptions. I was able to find some excellent problems in the literature, and identified excellent ideas for creating new problems. For once, Garfield and Konold's idea to construct two-part problems appealed to me. The first part asks a question and the second part calls for justification of the answer given in the first part. As all teachers know, a correct answer to a question does not necessarily imply proper understanding of a principle or concept and may be given for a wrong reason. Asking for justification in the second part reduces the likelihood of overlooking a misconception.
It also occurred to me that I could increase the utility of the test by creating pairs of distracters consistent with various misconceptions. In this way a single question could be used to identify different misconceptions.

It must be noted that about half of the items on this instrument are original. The other half are similar to those designed by Konold (1990), Garfeild (1998), and Hirsch and O'Donnell (2001). Still, practically all the borrowed items were modified to align them with the purpose and format of this instrument. The modification normally included adding or replacing distracters, rephrasing the problem or editing it.
What follows is a typical item with the explanation of the role of its elements.

#1. If a fair coin is tossed 6 times, which of the following *ordered* arrangements of heads and tails is LEAST likely to occur?
a. HTTHTH
b. HHHHHT
c. HTHTHT
d. *b* & *c* are equally unlikely and both less likely than *a*.
e. All of the arrangements are equally unlikely.

Which of the following best describes the reason for you answer to the preceding question?
a. Impossible to tell which arrangement will not occur, any of the arrangements could fail to occur
b. In this situation each of the ordered arrangements has the same probability of occurring or not occurring
c. Tossing of a coin is random, and random events always have the same probability of occurring or not occurring.
d. There ought to be about the same number of heads and tails
e. Since tossing of a coin is random, the coin is unlikely to alternate between heads and tails.

The distracters in this item reflect one of the three misconceptions: the correct answer **e**. to the first part when coupled with **a**. on the second part, serves as an indication of the outcome orientation misconception; students who choose it interpret *least likely* as *will not occur*. The same correct answer to the first part when accompanied with **c**. suggests the equiprobability bias – all random events are equally likely. Answer **b**. grouped together with **d**. is indicative of the misconception of representativeness, in particular, the belief that a sample should reflect the expected 50% - 50% distribution of heads and tails. And finally, the choice of **c**. for the first part along with **e**. for the second part is consistent with another facet of represenativeness, in particular, that there should be no order in random events, and that *each* outcome must reflect that randomness. Other questions in this instrument called the Probability Reasoning Questionnaire (PRQ) were constructed in a similar manner and with similar objectives in mind.

To accomplish the third purpose – investigate approaches and strategies used by college professors to resolve the misconceptions – I recruited 13 instructors from 5 different colleges and universities in the U.S. Three different levels (tiers) of participation were established. All three tiers involved the administration of the PRQ as a pretest and posttest. The pretest was administered before an instructor began teaching probability in the course. The posttest was administered at the end of the semester. Instructors who chose to participate at tier I were teaching the concepts but did not address any of the misconceptions directly in their teaching. Tier II participants used instructional materials prepared by the investigator and designed with the aim of attacking the misconceptions directly. This level of commitment also involved maintaining a dialogue with the investigator over the course of the semester about various aspects of implementation. Yet another level of participation (tier III) involved the creation by the participants of their own instructional materials and using them along with the materials designed by the investigator or instead of them. This level also implied an active dialogue with the investigator on a regular basis, as well as monitoring students' progress and fine-tuning the instructional intervention. Some class sessions of Tier III participants were observed by the investigator. Distinctions between the tiers are summarized in a table:

Three Tiers of Participation in the Experimental Phase.

I	II	III
Features common to all three tiers • All instructors motivated students to participate in the study • All instructors administered the pretest and posttest • All instructors were interviewed by the investigator		
	Features common to tiers II and III only • Targeted misconceptions directly in instruction • Maintained an active dialogue with the investigator • Completed lesson report forms	
Features specific to individual tiers		
• Used formal methods of instruction; did not target specific misconceptions directly in instruction	• Used only instructional materials prepared by the investigator to target the misconceptions	• Designed and used their own instructional materials in addition to those prepared by the investigator • Some class sessions were observed by the investigator • Had more and deeper discussions with the investigator about the various aspects of the study than tier I or II participants

A typical assignment employed by one of the instructors in the treatment of misconceptions is presented below.
The idea for this assignment was borrowed from a paper by the Canadian researcher Fast (2001). The object of the assignment was to help students unlearn representativeness.

Discussion situation 2. Best chance of winning.

Misconception treated: reperesentativeness
Link to important concepts: law of large numbers, independence, binomial distribution

Placement in the course: when discussing the law of large numbers
Time: 20 minutes
Organizational format: whole class or small group discussion
Development: You finished first in a chess tournament. You are confident that you are indeed the best player. However, the rules require that you must compete in a playoff against the student who finished second in the tournament. What would you prefer?
a) a 5-game series, b) a 9-game series? c) doesn't matter
Justify your choice.

Students who rely on the representativeness heuristic say that it does not matter. They believe that a 5-game series will reflect their superiority as well as a 9-game series.
Even when the instructor took time to compute the probabilities and demonstrates that the probability of winning a 9-game series is higher, many students remained unconvinced.
Some relied in their judgment on superficial factors and asserted that a 5-game series is better because they would be less likely to 'get tired'. Interestingly many of these same students rejected a one-game playoff as too risky. Experience suggested to them that you do not win **every** game even if you are the best. So they would prefer a 5-game playoff to a 1-game playoff. It is the extremity of the situation that made a correct choice more obvious. The instructor used this correct response as an anchor for the statistically correct concept that *a large sample is more likely than a small sample to reflect the characteristics of the population*. Flukes are more likely in smaller samples.

Results:

College professors' dominant views:

- The study of probability is important to all college students;
- A large proportion of college student have misconceptions about probability;
- Some instructors have misconceptions about probability;
- Students' misconceptions about probability need and can be addressed in introductory college statistics during the time normally allocated to the study of probability;
- Misconceptions about probability have a negative impact on students' ability to comprehend inferential statistics;
- Effective strategies exist for addressing misconceptions of probability;
- Instructors, in general, do not address students' misconception of probability in class;
- Teacher-centered instruction is more effective in resolving students' misconceptions of probability than small group cooperative learning, peer tutoring, independent study, etc…

Approaches and strategies used by participating instructors:

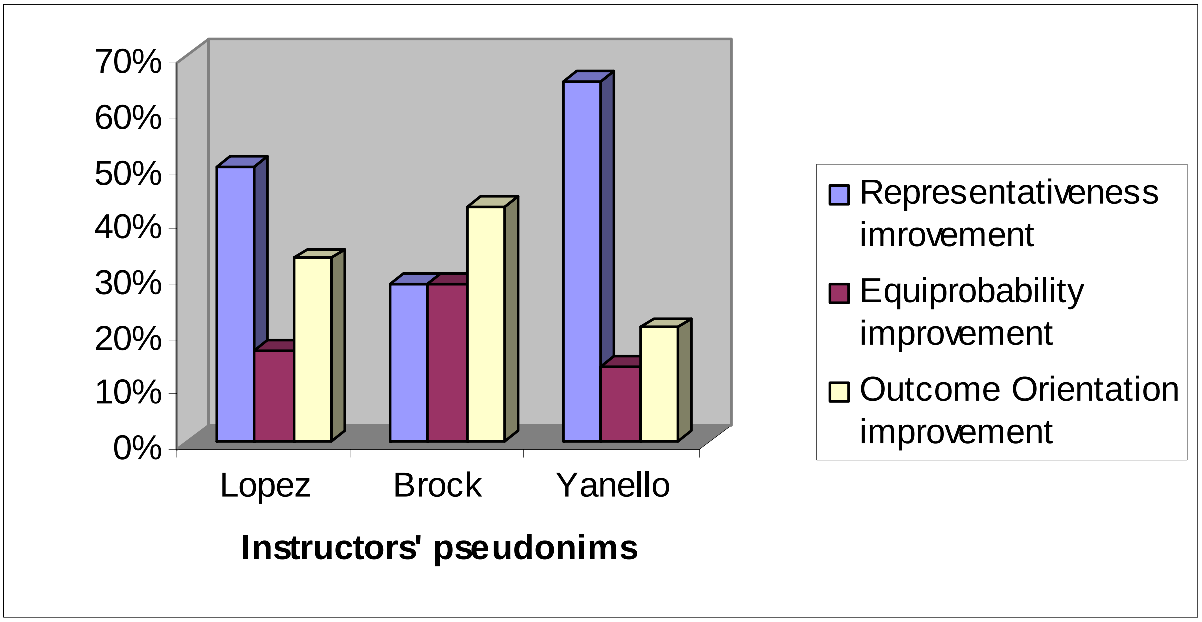
- Discussed misconceptions during lectures
- Gave students written assignments that addressed misconceptions
- Had students explore computer-generated simulations of random processes and observe patterns

- Had students work in small collaborative learning groups comparing predictions based on different models with outcomes of hands-on experimentation
- Used analogies and anchors

The effects of teaching:

- Most tier II and III instructors were successful in facilitating the resolution of students' misconceptions
- Some tier I instructors were successful
- Most tier I instructors were not successful
- Successful instructors did well on representativeness and outcome orientation. Did not obtain major gains on equiprobability

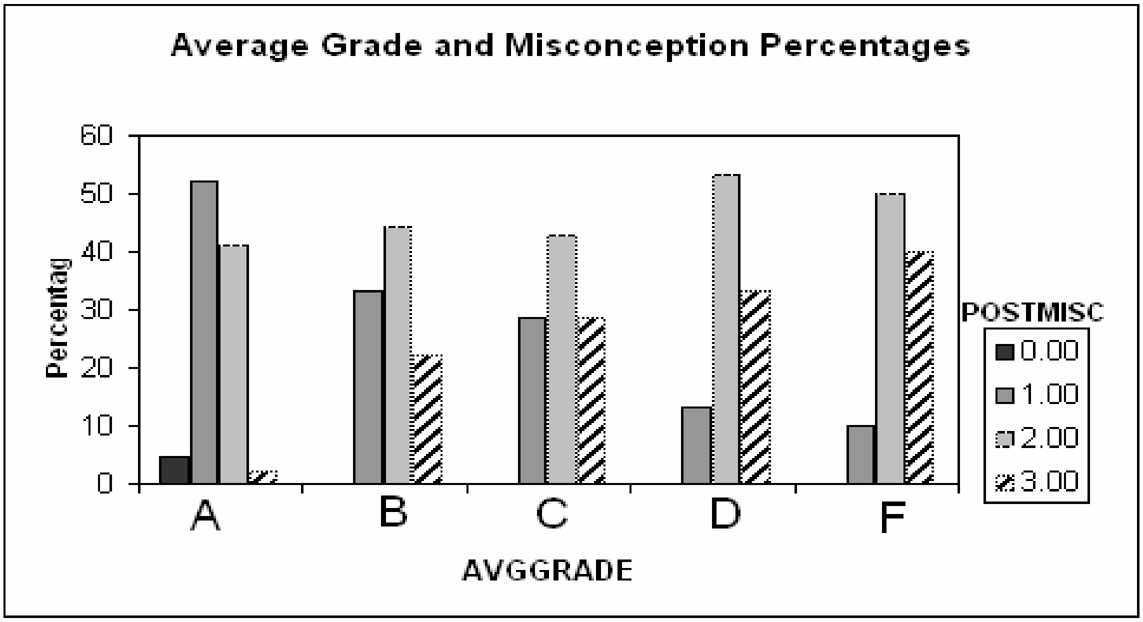
Percent improvement on individual misconceptions by successful instructors.



The equiprobability bias is difficult to unlearn for a variety of reasons. In addition to understanding chance in general, students must have the ability to quantify chance events and compare probabilities numerically. If students do not have sufficient experience with counting and unequally likely outcome models, they are prone to retain this misconception. I looked at the probability models considered in adopted textbooks and noticed that most of them are equally likely outcome models. A larger number of 'unequally likely outcome models' need to be explored by students before one could expect a significant number of them to unlearn the equiprobability bias.

Association between misconceptions of probability and achievement in statistics:

Students' achievement in statistics is negatively correlated with misconceptions about probability. (Achievement was measured by the grade assigned by individual professors.)



This is one of the most remarkable results. How can it be accounted for? It is tempting to say that misconceptions of probability affect students' ability to understand inferential statistics. There is some evidence to that effect but it is mostly indirect (for example, the opinions of some instructors). In this study I did not have access to students' written or oral documents, except for the PRQ. Caution must be exercised in interpreting these results. As we profess to our students: correlation does not imply causation. There may be a third variable at play here!

Professional development:
In a follow up study a colleague of mine Annette Gourgey and I (Khazanov & Gourgey, paper submitted for publication to JSE in fall of 2005) asked college statistics instructors if they would be interested in professional development in the area of probability and misconceptions about probability. The response was overwhelmingly positive. We are currently attempting to establish what the professors' needs are and how to address them best.
Based on the information we have now, it appears that a useful program could concentrate in these areas: 1) provide general information about misconceptions of probability and help instructors unlearn their own misconceptions 2) help instructors understand their students' misconceptions and how they affect students' learning of probability and statistics and 3) show instructors effective approaches, strategies, and techniques so they could improve the teaching of probability and facilitate the resolution of misconceptions in their students. The need for additional topics may surface as we explore this area further.

Final remarks:
This study was exploratory in nature. It may be extended in many different directions. Possible extensions include:

- revising survey questions and administering the survey to a larger sample of college professors
- identifying misconceptions that hinder students' ability to understand statistical inference

- designing assessment instruments for identifying other common misconceptions of probability
- developing new instruction materials to address students' misconceptions of probability

References:

Fast, G. The Stability of Analogically Reconstructed Knowledge in Secondary Mathematics Students. *Canadian Journal of Science, Mathematics, & Technology Education Volume 1, Number 2, April 2001.*

Garfeild, J. "The Statistical Reasoning Assesment: Development and Validation of a Research Tool," in *Proceedings of the Fifth International Conference on Teaching Statistic,* Ed. Pereira-Mendoza, Voorburg, The Netherlands: International Statistical Institute, 781-786 (1998).

Kahneman, P., Tversky, A. On the Psychology of Prediction, in *Psychology Review,* 80, pp. 237-251 (1973).

Hirsch, L.S. & O'Donnell, A.M. (2001) Representativeness in Statistical Reasoning: Identifying and Assessing Misconceptions, *Journal of Statistics Education*, V9N2 (2001).

Konold, C. Test Items to Identify Misconceptions in Probability and Statistics, unpublished document, University of Massachusetts, Scientific Reasoning Research Institute, Amherst. (1990).

Konold ,C. "An Outbreak in Belief in Independence?" in *Proceedings of the 11th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, (Vol. 2), eds. G. Goldin, C. Maher, and B. Davis. Rutgers, NJ: Rutgers University Press, 203-209 (1989).

Konold, C. Issues in Assessing Conceptual Understanding in Probability and Statistics, in *Journal of Statistics Education,* 3(1), http;//www.amstat.org/publications/jse/ (1995).

Lecoutre, M. P., Rezrazi M. Learning and transfer in isomorphic uncertainty situations: The role of the subjects cognitive activity. *In* L. Pereira-Mendoza, L. Seu, T. Wee, W. K Wong (Eds.): *Statistical Education – Expanding the network, Proceedings of the fifth International Conference on Teaching of Statistics, Vooburg, Netherlands: ISI Permanent Office, Vol. 3,* 1105-1111 (1998).