Some Connections Between Mathematics and Music

Joe Browne
Onondaga Community College
Syracuse NY 13215
(315) 498-2649
brownej@sunyocc.edu

Abstract: An explanation of harmonics, scales, temperament (Pythagorean, Just, and Equal) and fret placement on a guitar is given using precalculus level mathematics.

Mathematics is music for the mind
and music is mathematics for the soul.

Stanley Jordan

Rest In Peace

I mourn the loss of
Arithmetic
Not the subject, which is alive and struggling;
I mourn the loss of the *word* "arithmetic" which has been replaced, in the minds of the general public, by the word "mathematics." It's probably a snobbery thing; "mathematics" sounds nobler. The result has been a growing misconception of what mathematics is.
The statement "2 + 2 = 4" is not mathematics.

Levels of Mathematics in Music

Arithmetic

- Note lengths (half notes, eighth notes)
- Time signatures ($\frac{4}{4}$, $\frac{6}{8}$)
- Intervals (third, octave, half step, whole step)
- Chords (triad, seventh, ninth)

Algebra (and Physics)

- Frequency, pitch (A = 440 hz)
- Scale (major = whole whole half whole whole whole half)
- Temperament (Pythagorean, Just, Equal)
- Instrument design (string length, hole or fret placement, physical dimensions)

Structure and creativity

- Form (rondo, theme and variations)
- Composing and arranging (within the constraints of a chosen form, style, or ability level)
- Fingering

Harmonics I

Frequencies for a vibrating string are governed by the following equation:

$$f = \frac{k\sqrt{T}}{L}$$

where

f = frequency
L = string length
T = string tension
k = a constant

By touching a vibrating string at its midpoint, it will automatically divide itself into two vibrating strings, each half the length of the original. This is called the first harmonic. Since the tension, thickness of the string, etc., remain constant, the vibrating frequency is doubled, producing a note one octave higher than the original.

A similar phenomenon occurs when we touch the string so as to divide it into thirds (second harmonic, tripling the frequency), fourths (third harmonic, quadrupling the frequency), etc. As the harmonics get higher, the sound also gets weaker.

At this moment we will focus our concentration on the first two harmonics. These produce notes an octave and an octave plus a fifth above the fundamental note. What we are to remember from this is that the frequency ratio for the interval called a fifth is 3:2.

Pythagorean Scale

In the Pythagorean Scale the interval of a fifth (say C to G) has a frequency ratio of 3:2. For example, if a note has a frequency of 60 hz, the note a fifth above has a frequency of 90 hz. The object of this exercise is to calculate the frequencies of one octave worth of Pythagorean scale, starting with C = 256. The notes a fifth apart, in order, are

C, G, D, A, E, B, F#, C#, G#, D#, A#, E#, B#.

So, to get the frequency for G, I multiplied 256 by 3/2 to get 384. Now, to keep the frequencies within one octave, i.e. between 256 and 512 (except for the last note), you will need to divide some of them by 2. Thus, to get the D below, I multiplied G = 384 by 3/2, which gives 576, and then divided by 2 to get D = 288.

B# (= C?)	
B	
A#	
A	
G#	
G	384
F#	
E# (= F?)	
E	
D#	
D	288
C#	
C	256

In music we normally say that B# and C are the same note. Hence the 13th note in our sequence above should be a C an octave up from the starting point. Moving an octave up exactly doubles the frequency. How does our B# compare with the octave C (= 512)?

Just Temperament

Worked out by Vincenzo Galilei (1525-1591) (father of Galileo).

Just temperament is based on the observation that the most pleasing sounding intervals are those for which the frequencies are ratios of small integers.

Do	Re	Mi	Fa	Sol	La	Ti	Do	Re	Mi	Fa
1	9/8	5/4	4/3	3/2	5/3	15/8	2	9/4	5/2	8/3

- All intervals are ratios of relatively small integers.
- The I, IV, and V triads (chords) have a pleasing 4:5:6 ratio.
  I triad = 1:5/4:3/2 = 4:5:6
  IV triad = 4/3:5/3:2 = 4:5:6
  V triad = 3/2:15/8:9/4 = 4:5:6
- The fifth based on Re (II) has the ratio 5/3:9/8 ≠ 3:2.
- Notes tuned "just" in one key are not tuned "just" in other keys.

A more complete scale in Just Intonation looks like this (based on C = 256) with a comparison to the Pythagorean frequencies.

Note	Just Ratio	Just Frequency	Pythagorean Frequency
C	2	512.0	519.0
B	15/8	480.0	486.0
A#	9/5	460.8	461.3
A	5/3	426.7	432.0
G#	8/5	409.6	410.1
G	3/2	384.0	384.0
F#	45/32	360.0	364.5
F	4/3	341.3	346.0
E	5/4	320.0	324.0
D#	6/5	307.2	307.5
D	9/8	288.0	288.0
C#	16/15	273.1	273.4
C	1	256.0	256.0

Equal Temperament

Promoted by Johann Sebastian Bach (1685-1750). The system existed much earlier, chiefly used by fretted instruments, the ancestors of the guitar.

Equal temperament is based on the notion that the ratio between the frequencies of any two adjacent notes in the 12 note scale should be the same. Let's call this ratio *r*. Thus, if we start with A = 440, we would multiply this by *r* to get the frequency for A#, multiply again by *r* to get the frequency for B, etc.

To move up one octave we would have to multiply by *r* twelve times to double the frequency. Thus, we have

$$r^{12} = 2$$

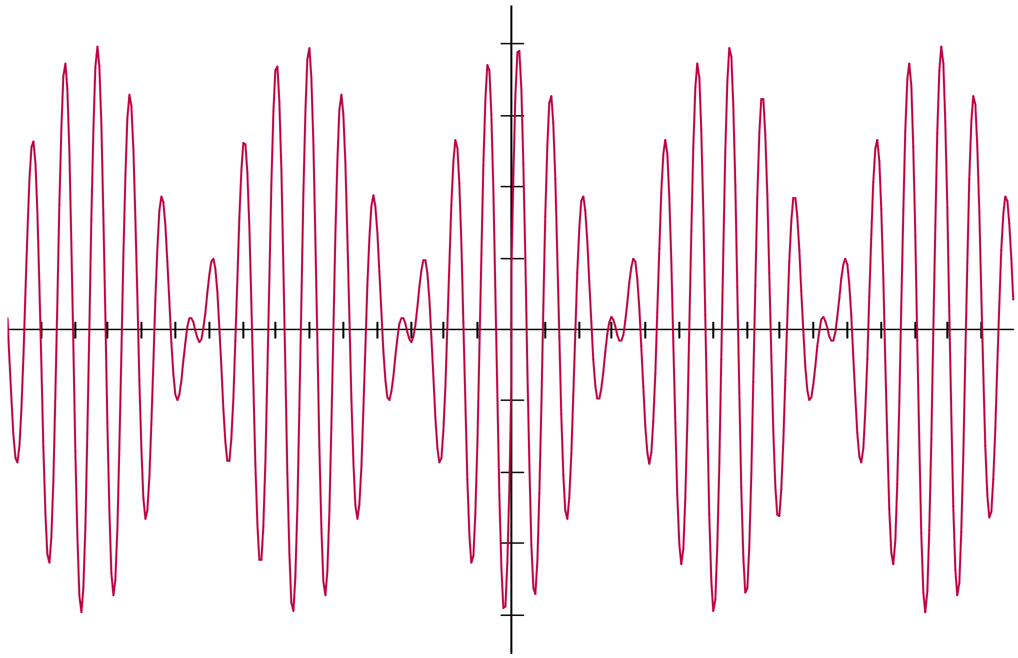
$$r = \sqrt[12]{2} \approx 1.05946\ldots$$

Characteristics of Equal Temperament

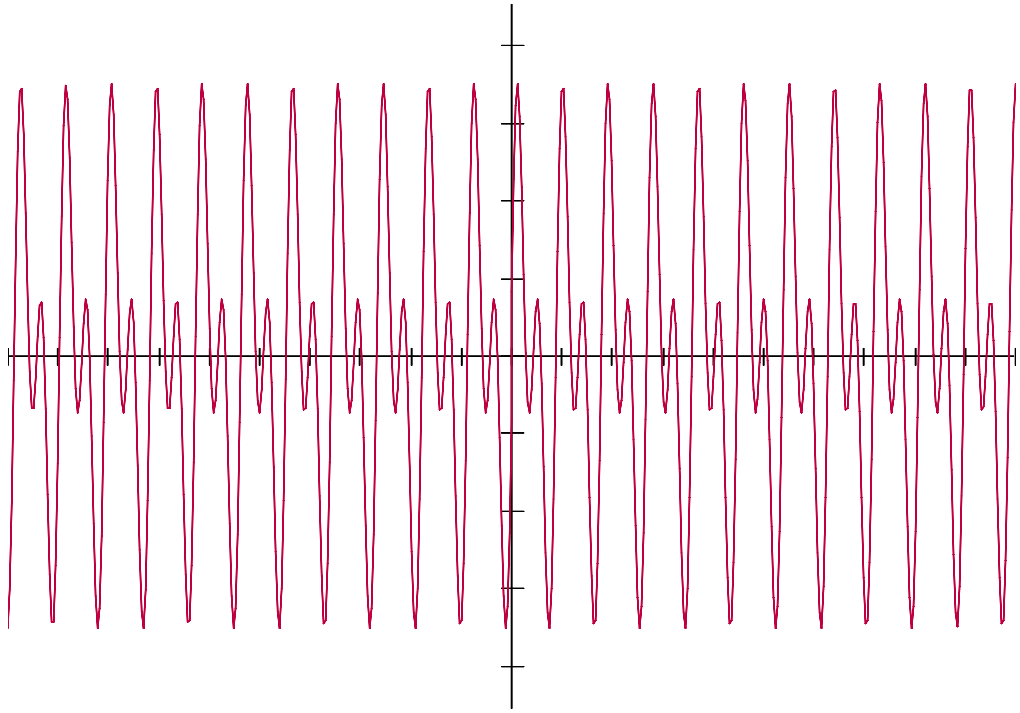
- Intervals in one key are exactly the same as any other key.
- Fifths have a ratio of $2^{7/12} = 1.4983...$ instead of 3/2.
- The only interval with an integer ratio is the octave (2/1).
- All intervals other than the octave have irrational frequency "ratios." That is, to a person quite sensitive in hearing these ratios and accustomed to just temperament, all intervals (except the octave) are out of tune.

Temperament Ratio Comparison

Syllable	Pythagorean	Just	Equal
Do	2.02729	2.000	2.00000
Ti	1.89844	1.875	1.88775
	1.80203	1.800	1.78180
La	1.68750	1.66666…	1.68179
	1.60181	1.600	1.58740
Sol	1.50000	1.500	1.49831
	1.42353	1.40625	1.41421
Fa	1.35152	1.33333…	1.33484
Mi	1.26563	1.250	1.25992
	1.20135	1.200	1.18921
Re	1.12500	1.125	1.12246
	1.06787	1.06666…	1.05946
Do	1.00000	1.000	1.00000



Sum of two waves with frequencies that are close, but not equal. Notice that they sometimes contribute to each other and sometimes cancel. A clearly audible pulsation or "beat" is produced.



Sum of the waves of two notes an octave apart (one has double the frequency of the other). While there is still support and cancellation, there is no throbbing "beat" and the notes sound clean and pure.

Harmonics II (Just Temperament)

E (sol)	1320	1320	1320	
	1210			
		1155		
C# (mi)	1100		1100	1100
B (re)	990	990		
A (do)	880		880	
G# (ti)		825		825
	770			
E (sol)	660	660	660	
C# (mi)	550			550
B (re)		495		
A (do)	440		440	
E (sol)	330	330		
C# (mi)				275
A (do)	220		220	
E (sol)		165		
A (do)	110			

Harmonics II (Equal Temperament)

Harmonics know about physics, but not about temperament. Thus, they continue to sound at two, three, four, etc., times the base frequency. The result is that they "miss" some of the notes we might hope they would hit.

E (sol) 1318.51	1320	1318.51	1320	
	1210			
		1153.70		
C# (mi) 1108.73	1100		1100	1108.73
B (re) 987.76	990	988.88		
A (do) 880.00	880		880	
G# (ti) 830.61		824.07		831.55
	770			
E (sol) 659.25	660	659.25	660	
C# (mi) 554.36	550			554.36
B (re) 493.88		494.44		
A (do) 440.00	440		440	
E (sol) 329.63	330	329.63		
C# (mi) 277.18				277.18
A (do) 220.00	220		220	
E (sol) 164.81		164.81		
A (do) 110.00	110			

Fret Placement on a Guitar

A glance at any guitar will reveal that the frets are not equally spaced. As you move "up" the fingerboard the frets get closer together. One old, traditional principle used by some luthiers in the past to locate frets was this.

- Divide the length of the string by 18 to determine the distance from the nut to the first fret.
- After placing that fret, divide the remaining distance by 18 to find the distance to the next fret.

This method is simple, but loaded with flaws.

1. The theoretical divisor isn't really 18. Repeating this process 12 times will not arrive at the midpoint of the string. (Note: the theoretical divisor is approximately 17.817. One luthier, in a published article, called this number the twelfth root of 2. Well, it does have something to do with the twelfth root of 2, it's $\frac{\sqrt[12]{2}}{\sqrt[12]{2}-1}$, but we should let this fellow make guitars and leave the arithmetic to others.)
2. Since each calculation depends on the result of the previous step, any error is repeated in each step and accumulates as the procedure is repeated.

Let's start from basic scientific principles: the equation governing frequency of a vibrating string:

$$f = \frac{k\sqrt{T}}{L}$$

where f = frequency
L = string length
T = string tension
k = a constant

We assume that tension, T, is constant, so we can simplify this to

$$f = \frac{c}{L} \qquad \text{or} \qquad L = \frac{c}{f}$$

where c is a constant. We want to place the first fret so that the frequency is one half step up,

$$f_{new} = \sqrt[12]{2}\, f_{old}$$

We can then calculate the effect on the string length, L.

$$L_{new} = \frac{c}{f_{new}} = \frac{c}{\sqrt[12]{2}\, f_{old}} = \frac{\frac{c}{f_{old}}}{\sqrt[12]{2}} = \frac{L_{old}}{\sqrt[12]{2}}$$

This is correct, but a little inconvenient. It gives us the location of the fret with respect to the bridge saddle. For reasons of "compensation" (we may get a chance to discuss this later) the bridge is not located in its exact theoretical position and the difference varies from string to string. As a result, frets are placed according to their distance from the nut.

So, let

x_i = distance from i[th] fret to nut, and
S = scale length (theoretical string length)

Then

$$S - x_1 = \frac{S}{\sqrt[12]{2}} \longrightarrow x_1 = S - \frac{S}{\sqrt[12]{2}} = S\left(1 - \frac{1}{\sqrt[12]{2}}\right)$$

For the second fret the string length is just divided by $\sqrt[12]{2}$ again, so we get

$$S - x_2 = \frac{S}{\left(\sqrt[12]{2}\right)^2} \longrightarrow x_2 = S - \frac{S}{\left(\sqrt[12]{2}\right)^2} = S\left(1 - \frac{1}{\left(\sqrt[12]{2}\right)^2}\right)$$

A pattern is now evident, and we get

$$x_i = S - \frac{S}{\left(\sqrt[12]{2}\right)^i} = S\left(1 - \frac{1}{\left(\sqrt[12]{2}\right)^i}\right).$$

Of course this sort of calculation is easily done in a spreadsheet, and several fret location calculators are found on the internet.

Problems:

1. I inquired as to the scale length (string length) of a guitar seen on ebay. The (rather confused) seller reported that the length of the fingerboard was 16". From the photograph I could see that the guitar had 19 frets. What was the scale length?

2. Maria has small hands and sometimes has troubles with reaching notes on the guitar. She has found that placing a capo at the first fret helps tremendously and wants to buy a guitar with that shorter scale length. If a standard guitar has a scale length of 650mm, what scale length should she look for?

Compensation

When a string is pressed against a fret, its tension is increased, making the frequency increase and the note sounds sharp. When constructing the guitar the maker must "compensate" for this effect by moving the saddle of the bridge slightly farther away from the nut. The amount of compensation depends on several things including the distance the string is displaced when fretted and the thickness and density of the string. Typically the amount of compensation is 2-5 mm and is different for each string. Steel string guitars usually need more compensation and you will see a visible slant of the saddle to accomplish the differences between the strings.

Bibliography

Cumpiano, William and James Natelson. *Guitarmaking: Tradition and Technology*. Chronicle Books, 1994. This book is a highly regarded volume on guitar construction which includes a discussion of temperament, fret spacing math, and an interesting geometric construction (compass and straightedge) for fret spacing. Cumpiano also maintains a very informative website: www.cumpiano.com.

Gadbois, Steve. "Math and Music." Presentation at AMATYC conference. 2003. Slides available at www.amatyc.org/publications/Electronic-proceedings/SaltLakeCity29/SLC-proceedings.htm

Middleton, Rik. *The Guitar Maker's Workshop*. Crowood Press, 1997.

Rusin, David. Professor at Northern Illinois University has an interesting website: www.math.niu.edu/~rusin/papers/uses-math/music.

Sloane, Irving. *Classic Guitar Construction*. Bold Strummer, Ltd.1964.

Tauber, Charles. "Basic Guitar Set-up 101." A nice article which includes a discussion of temperament on the web at www.guitarnotes.com/notes/noteget.cgi?basic_guitar_setup.

Tomes, Ray. "Harmonics, Music, Pythagoras & the Universe." Article on website in New Zealand: http://ray.tomes.biz.