Calculating with Calculi: the Counting Board and Its Use in Reckoning in Medieval Europe

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Abstract: Attendees will learn some of the history of writing numbers and practice calculation algorithms used in the Middle Ages, using counting boards and counters called 'calculi.' The featured algorithm is one for multiplication that does not require knowing a multiplication table.

History

No one really knows the origin of the abacus or its early history. We know that a form of the abacus, the counting board, dates from at least the fourth century BCE. There is one example, the so-called Salamis counting board, found on the Greek island of Salamis that can be dated to that time. Other than that example, the early history of the abacus is conjectural. Forest Mims, III claims that "[t]he first abacuses were noting more that flat trays sprinkled with a thin coating of fine dust or fine sand. Various number symbols could be easily marked in the dust with a pointed stick and erased with a finger. The semitic word for dust is *abq* and some people think this is where the word *abacus* comes from." [1, p. 6]

There is also a Roman form of the abacus which is more like the Oriental abacus we are used to today. It consisted of a slotted plate of brass with brass pegs that moved in the slots.

In medieval Europe, the form of the abacus used was a version of the counting board. It was used up to the 16th century, when the pen and ink calculations gradually took over. This shift in methods was linked inextricably to the shift from the use of Roman numerals to the use of Hindu-Arabic numerals, known at the time as Arabic figures.

The Arabic number system was first noted by scholars in western Europe in the 11th century. Their use spread extremely slowly. In 1299 an edict in Florence forbade bankers to use Arabic figures. In 1348 the University of Padua mandated that books for sale have their prices marked in "good plain letters," i.e., Roman numerals.

In fact, there is a cultural artifact that comes from this struggle: The word 'cipher' means the Arabic zero (from the Arabic word 'sifr'). By extension, it came to mean any Arabic figure. It also came to mean a secret code, giving us an idea how Arabic figures were regarded!

Some 13th century financial documents used Arabic figures, and the calculations using them were explained in books in 1385 and 1480. The first coin to use a date in Arabic figures was a Tyrolian silver coin, in 1486. Coins in Britain did not use Arabic figures in dates until after 1548.

One very curious phenomenon was the almost indiscriminant mixture of the two systems as in MCCCC94. The combination of the two systems continued into the 19th century. Arithmetic books used Roman numerals to explain the Arabic system as late as 1793. [3, Chapter II]

Slightly after the shift to Arabic figures started, the shift from counting boards to pen and ink calculations started, continuing until about 1800. [3, Chapter III]

The conflict is shown in an illustration from a book by Gregor Reisch dated 1504: The central figure is Dame Arithmetic watching a competition between Boethius, using pen and the new method, and Pythagoras using the counting board. Dame Arithmetic is shown favoring the new method. Images of this illustration are available at the following websites (as of 11/4/2005):
http://singapore.cs.ucla.edu/LECTURE/section3/scan-71.gif
http://uwacadweb.uwyo.edu/numimage/Geometry.gif
http://www.chicagocoinclub.org/projects/PiN/jet2.gif
http://www.uta.edu/english/TAR/officeweb/arithmetic.jpg
http://www.es.flinders.edu.au/~mattom/science+society/lectures/illustrations/lecture6/margarita.html

Another image containing a counting board is bound in the website of the *Museum Judengasse* (Museum of the Jewish Ghetto) in Frankfurt-am-Main. It shows a farmer and a Jewish moneychanger at a counting board from a woodcut, Augsburg 1531. The image is available at http://www.judengasse.de/ehtml/Z107.htm

The Medieval Counting Board

The form of the counting board used in medieval Europe is very simple to understand if one understands the Roman numeral system, which is an example of an *additive number system*:

I represents one,	C represents one hundred,
V represents five,	D represents five hundred, and
X represents ten,	M represents one thousand.
L represents fifty,	

The counting board itself consists of a set of horizontal lines crossed by one or two vertical lines (See Fig. 1). The numbers were represented using small counters, often actually small pebbles. In fact, the Latin word for pebble is *calculus,* which is the root of our word calculate. Early in the use of the counting board, calculations were done using these *calculi*. Later the reckoners used counters or 'jettons,' which were made of metal and decorated. [3, Chapter III]

A counter placed on the horizontal line closest to the user represents one, and the succeeding lines represent tens, hundreds, thousands, etc. For five, the user places a counter between the ones line and the tens line, and fifty and five hundred are represented similarly. For example, the number in Figure 2 is MMDCCCCLXXVIII or 2,978.

A slightly different form of the counting table was the 'Exchequer table.' The markings on the Exchequer table came in a number of different forms, the most interesting for my purposes being shown below (Fig. 3). This table uses the European monetary system: £/s/d/f for 'Libra,' pound;

'solidus,' shilling; 'denarius,' penny; and farthing. The conversions are: £1 = s20, s1 = d12, d1 = f4. [3, Chapter III]

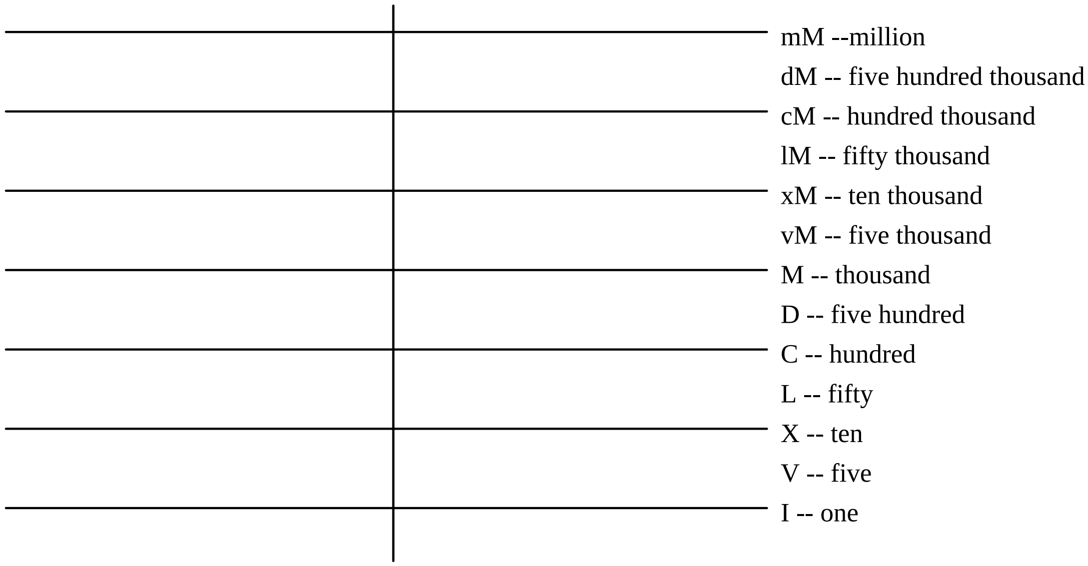

Figure 1: Counting Board

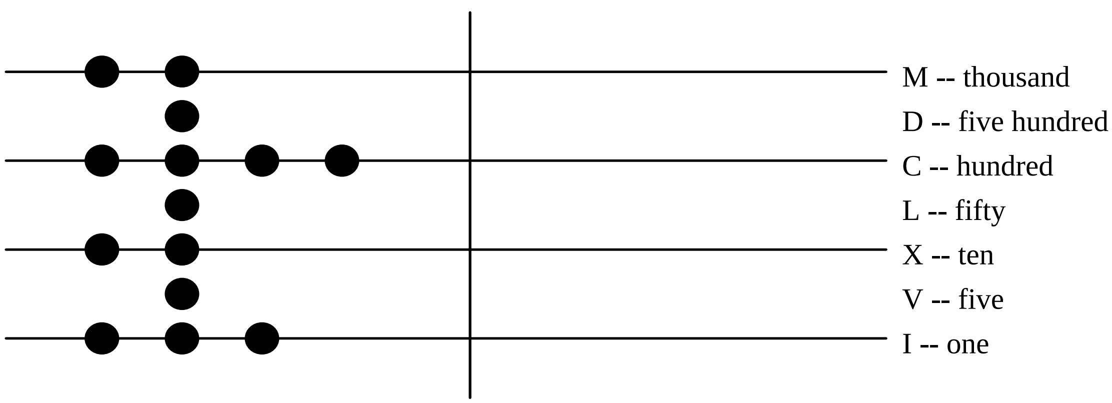

Figure 2: MMDCCCCLXXVIII or 2,978

Calculations

Now we will turn our attention to the actual methods of calculation used on a counting board. For our purposes, we will use the simple form shown first.
Note: Much of the following development is adapted from [2], Chapter 2.

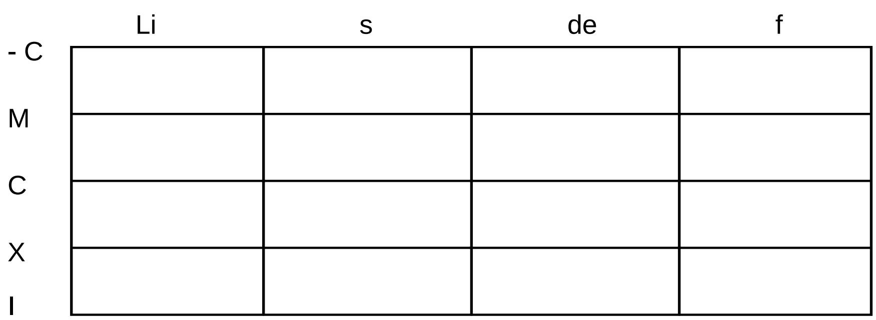

Figure 3: Exchequer Table

Addition

Addition on a counting board is relatively straightforward. We will use a counting board with two areas for numbers. The problem, written in modern form is: 4,284 + 2,533. In Roman numerals, the numbers to be added are MMMMCCLXXXIIII and MMDXXXIII. The first number will be placed in the left area, and the second in the right area. The sum will be placed in the left area.

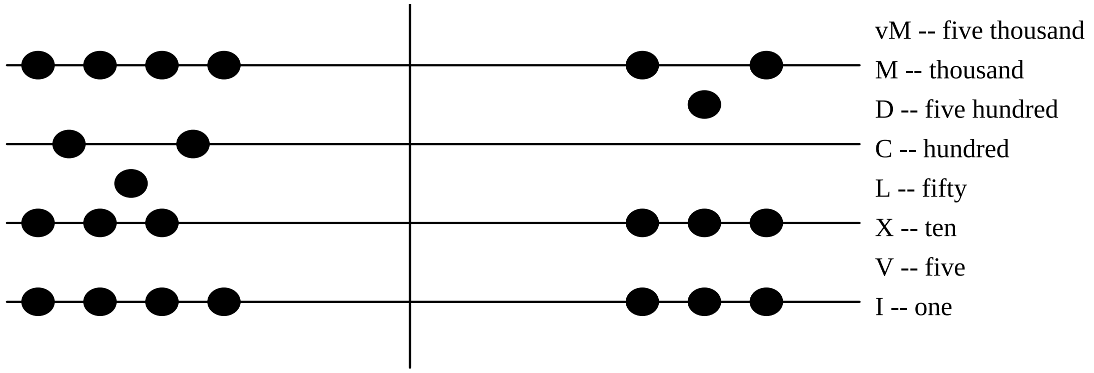

Figure 4: Addition

Adding the two numbers is really quite easy. Move the counters from the right to the left, and then replace five counters on a line with one counter above the line and two counters above a line with one counter on the next line up. The two states of the table are shown below. The circles in the first diagram indicate counters that have been moved.

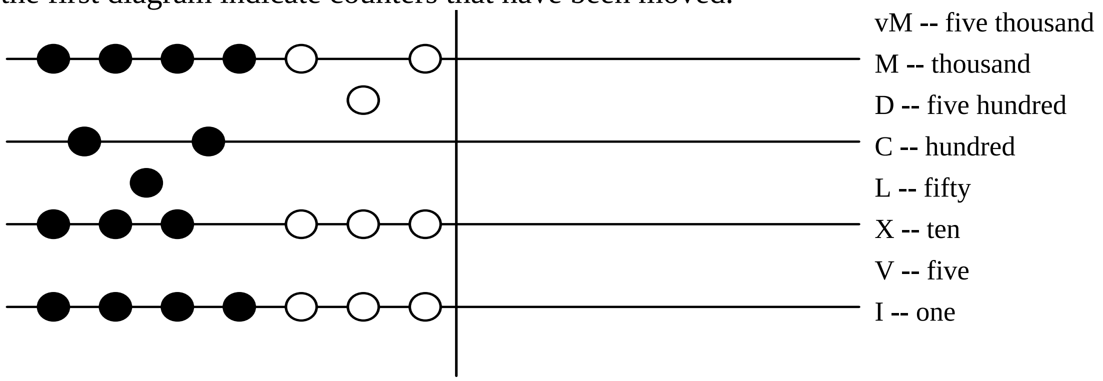

Figure 5: MMMMCCLXXXIIII and MMDXXXIII

Reading the sum off the second column: vM MDCCCXVII, which translates to 6,817 (Fig.6).

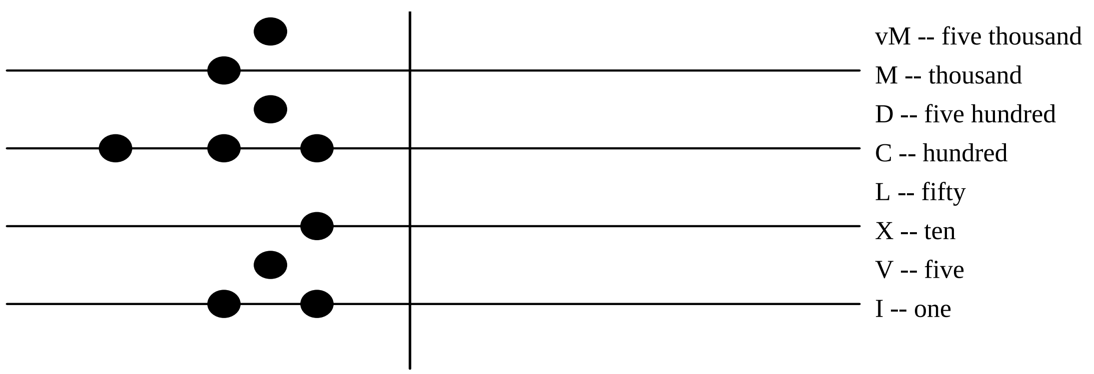

Figure 6: The sum, vM MDCCCXVII

Subtraction

Subtraction is also straightforward, in the sense that one is simply taking counters away. In order to make this possible, however, there is some trading down of counters that must be done before the calculation proper. For our example we will use 9,202 - 3,827, which translates into Roman numerals as MMMDCCCXXVII is to be taken from vM MMMMCCII(Fig. 7). The minuend and the difference will be placed in the left area and the subtrahend will be placed in the right area.

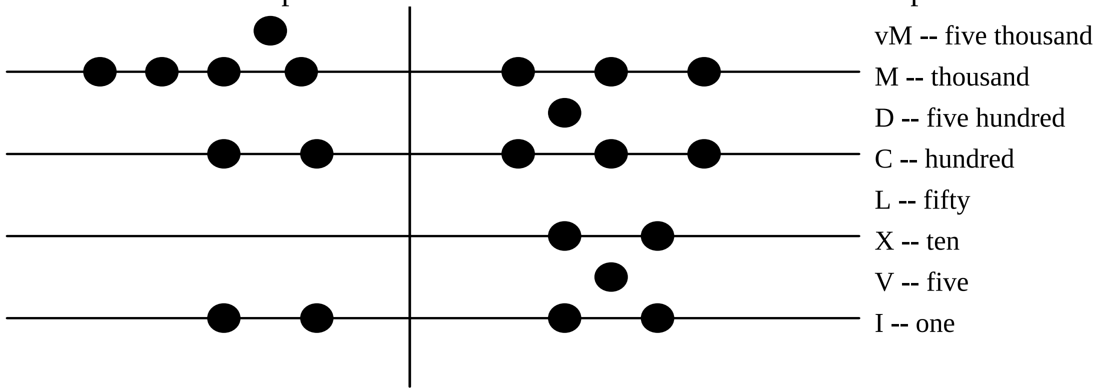

Figure 7: 9,202 - 3,827

The board after the borrowing step is (new counters are circles)

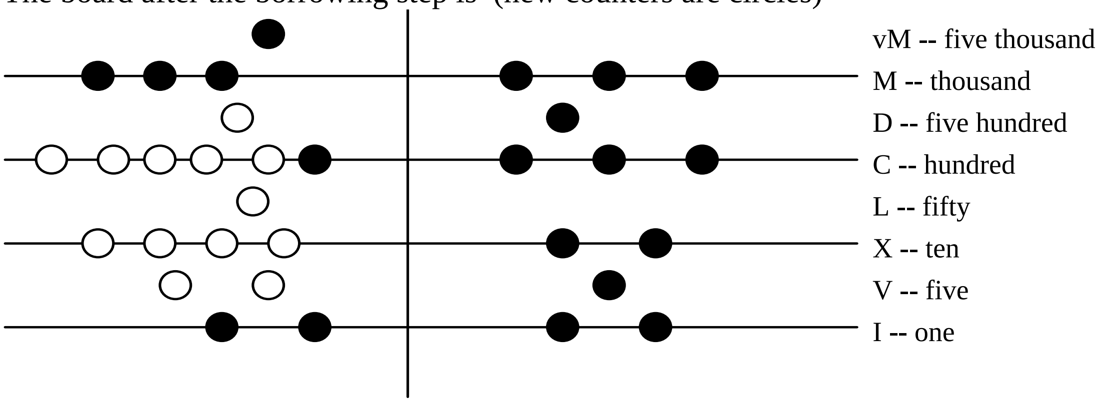

Figure 8

Now it is simply a matter of removing the counters on the left that correspond to the counters on the right. This yields (Fig. 9)

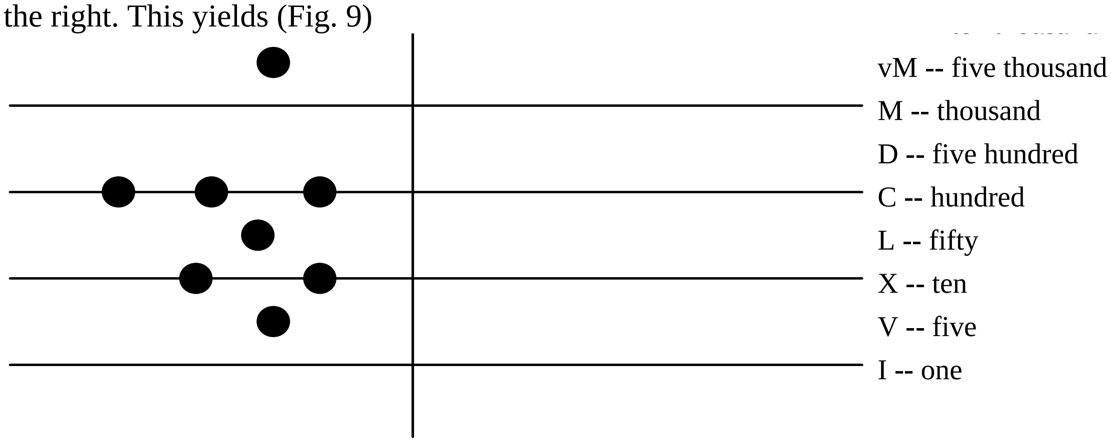

Figure 9: vM CCCLXXV

The result is vM CCCLXXV which translates to 5,375.

Multiplication

Addition and subtraction are fairly easy to grasp, as it is mostly a matter of counting. Multiplication on a counting board, like multiplication using Arabic figures, is more complicated. We will consider here two methods. Note that for multiplication and division we will use a table with three areas for numbers.

First Method

The first method is very similar to the pen and ink method, in that we use the counting board as a place to keep track of sums. This requires us to know the multiplication table.

The example we will use is 84 × 926. In Roman numerals this means LXXXIIII is to be multiplied by DCCCCXXVI. To set this up we will place the smaller number in the left space and the larger number in the middle. The result will appear in the right space.

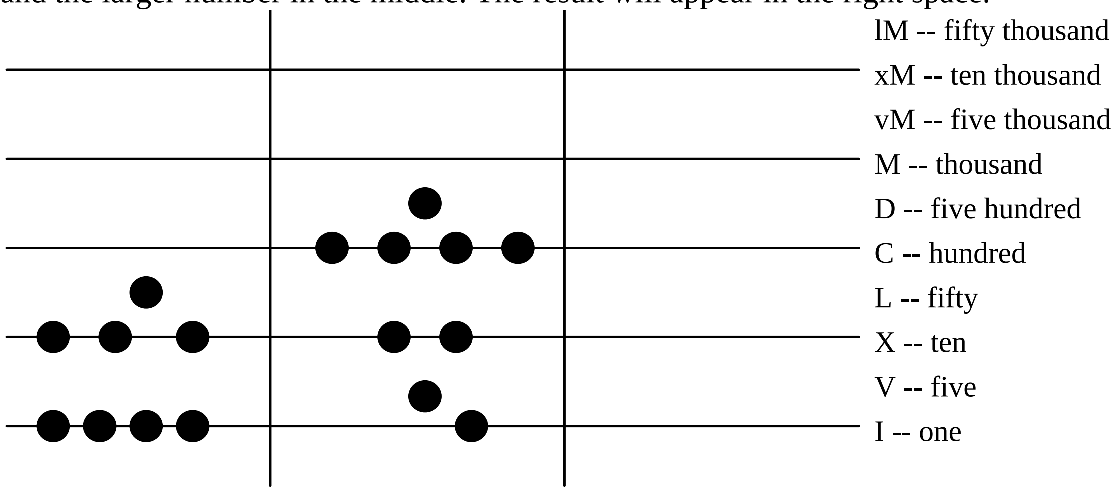

Figure 10: LXXXIIII is to be multiplied by DCCCCXXVI

We will start with the bottom line and the space above it. IIII times VI is XXIIII. Since the lines are both on the bottom, we will place the result starting on the bottom line thus:

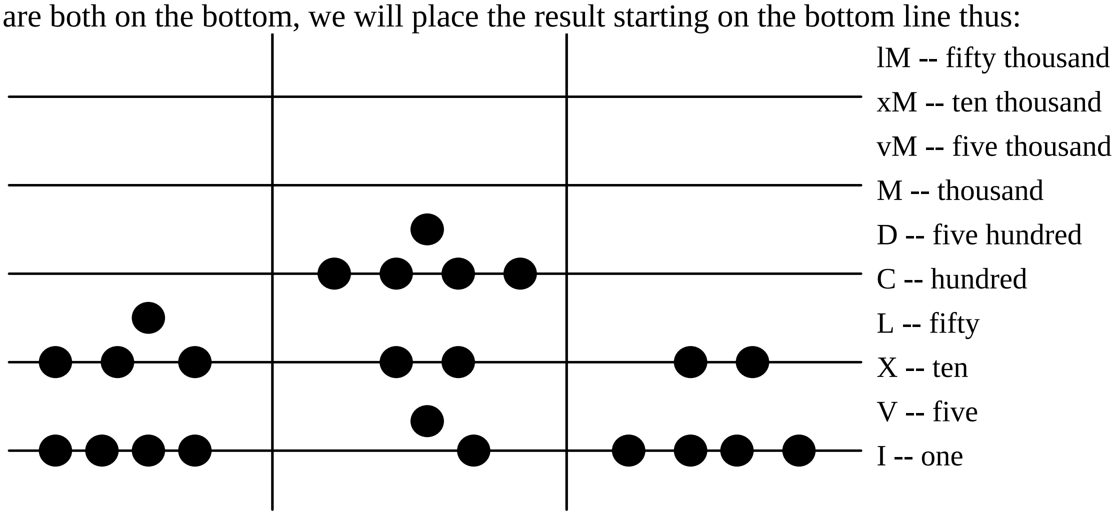

Figure 11

Now take the IIII and the left times XX, which yields LXXX. Add this to the board on the right, yielding

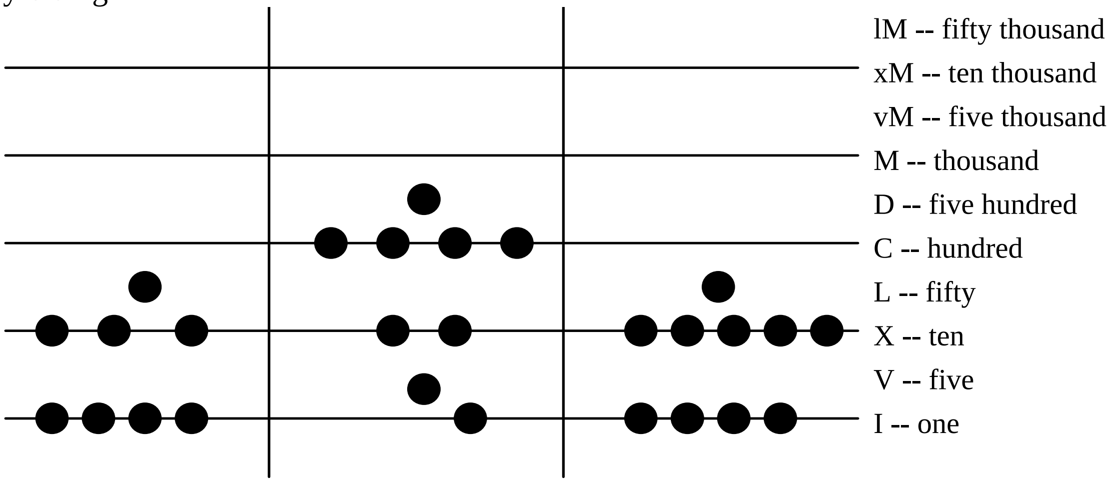

Figure 12: LXXX

Since there are five counters on the line we replace them with one above the line, yielding two above the line, which we in turn replace with on on the line above that.

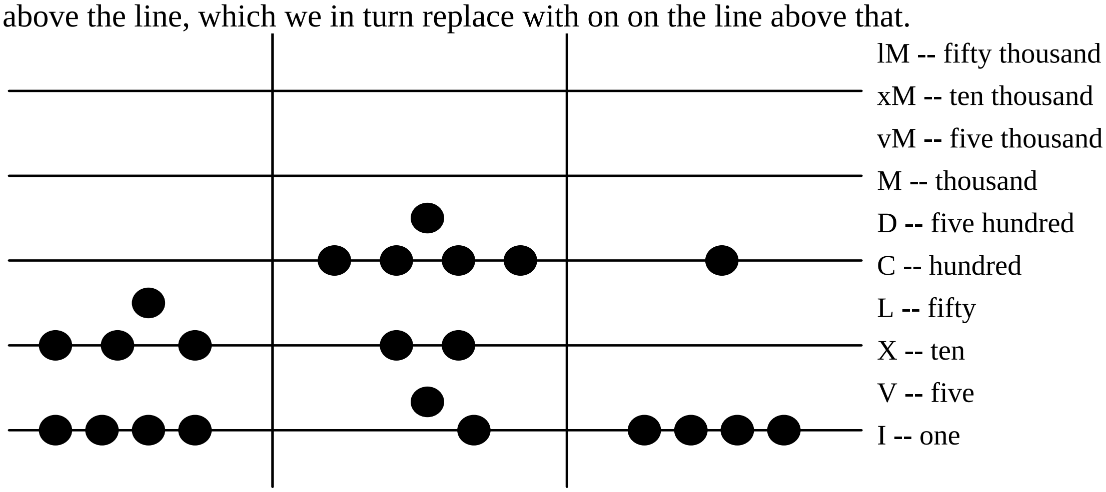

Figure 13

Lastly, IIII times DCCCC is MMMDC, which we place on the board. It needs no trading.

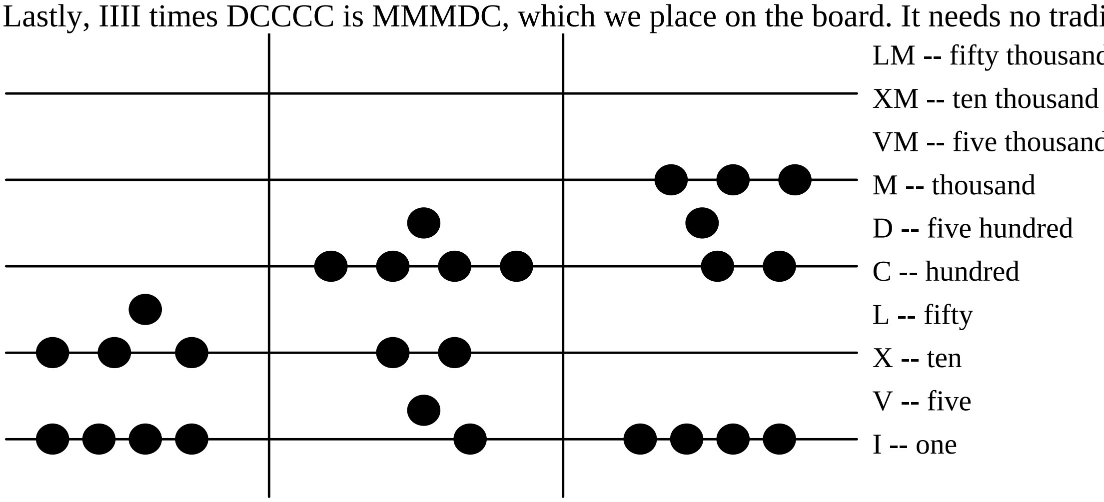

Figure 14

We now repeat the process with LXXX and the number in the middle. LXXX times VI is CCCCLXXX. Place the counters on the board

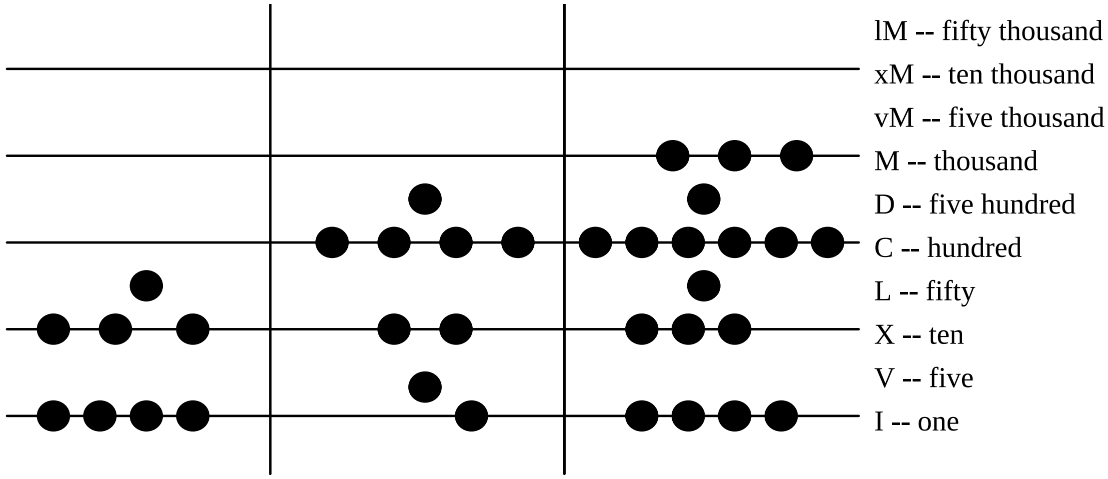

Figure 15

Trading counters yields

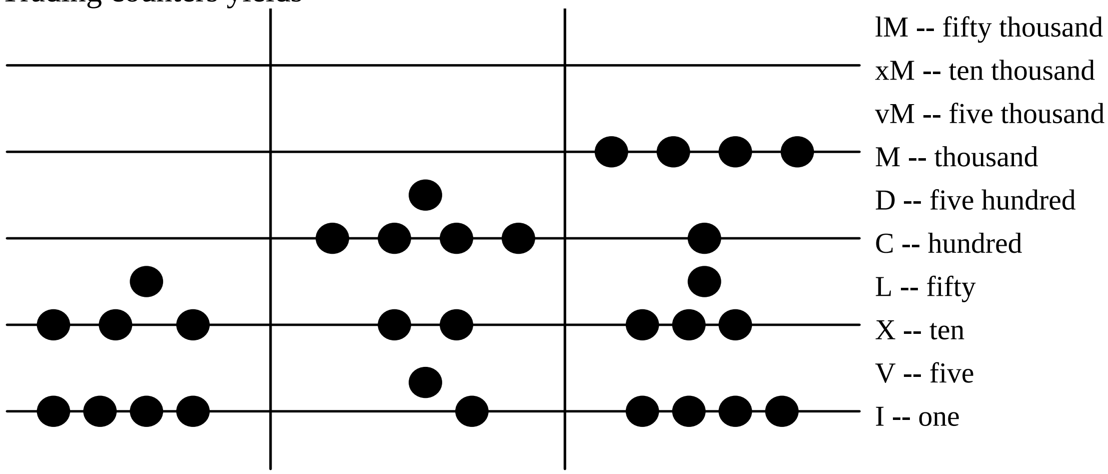

Figure 16

Now, LXXX times XX is MDC. We put this on the board

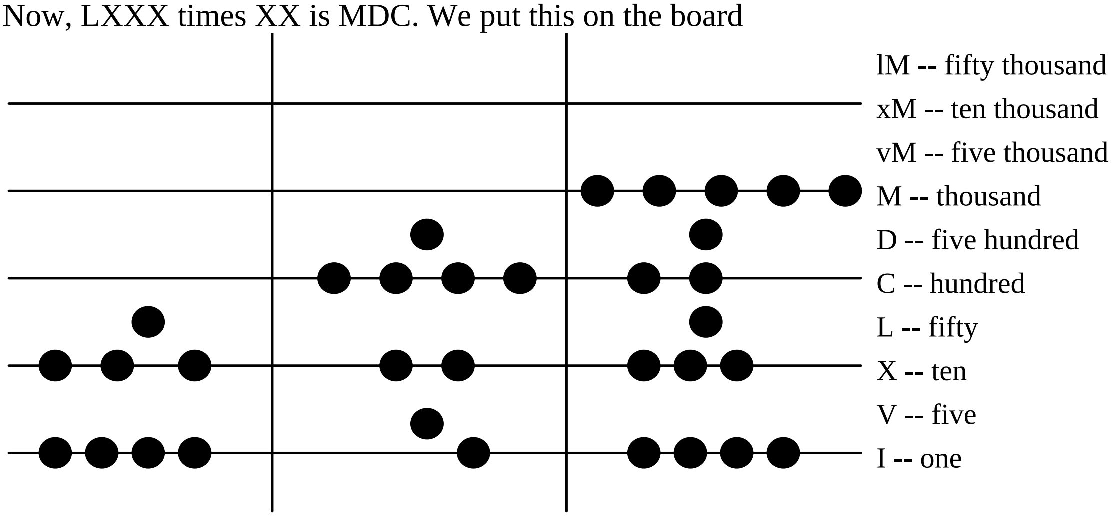

Figure 17

and trade counters to get

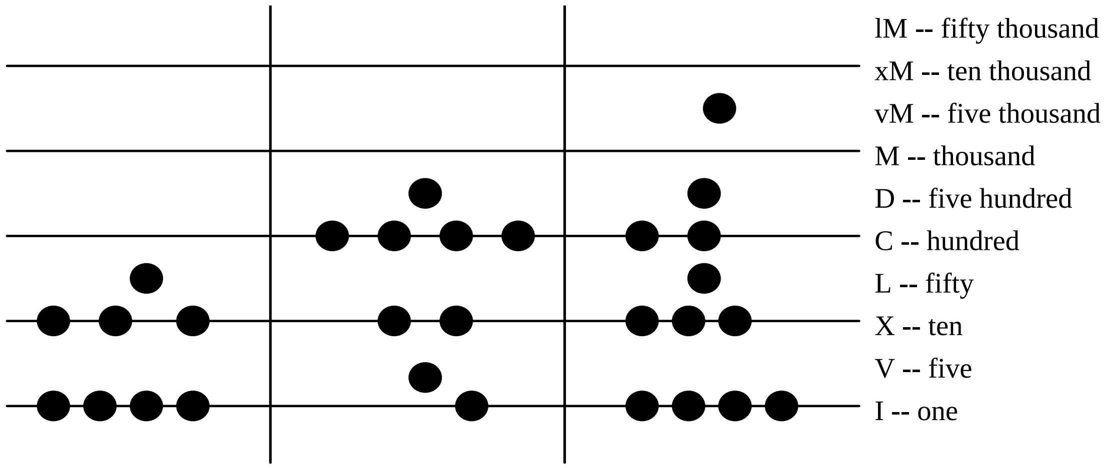

Figure 18

Finally, LXXX times DCCCC is lMxMxM MM. We put this on the board and get the final result

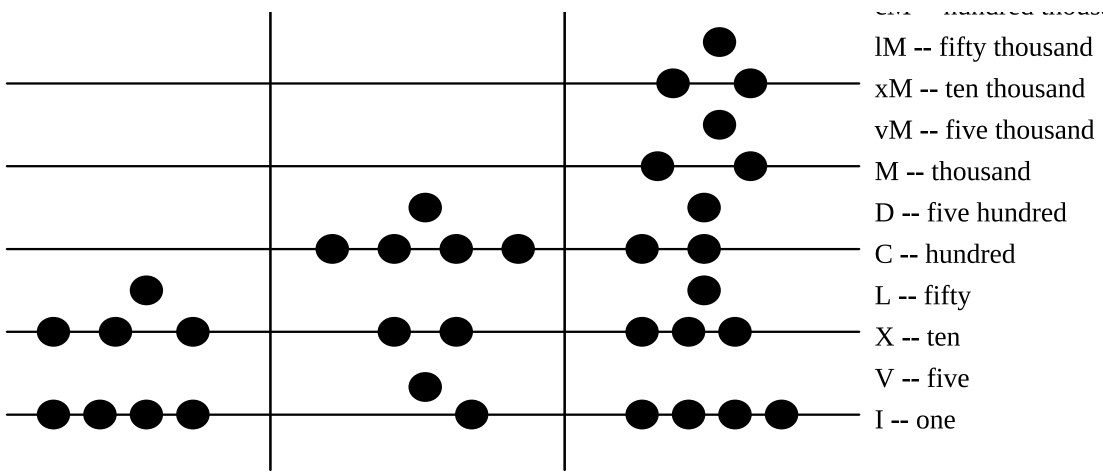

Figure 19

The final result is lMxMxMvMMMDCCLXXXIIII, which is 77,784 in Arabic figures.

Second Method: Duplation and Mediation

A different method for multiplying on a counting board, which was actually more common, was called *duplation and mediation*. This method has a distinct advantage, in that it can be performed by someone who does *not* know the multiplication table. It requires only the ability to cut one number in half and then double the other number. This is easy on a counting board.

As our example we will perform the multiplication 84 × 327 or LXXXIIII times CCCXXVII. We again use the board with three areas and put the smaller number on the left (Fig.20).

The method of duplation and mediation uses three basic steps:

1) Check to see if the number on the left is odd or even. If it is odd do step 2 followed by step 3. If the number is even go directly to step 3.

2) Add counters equal to the number in the middle to the space on the right, and remove one of the ones counters from the number on the left.

3) Halve the number on the left, and double the number in the middle. If there are any counters left in the left space, continue with step 1.

Starting from the original setup, we halve the left number and double the middle number.

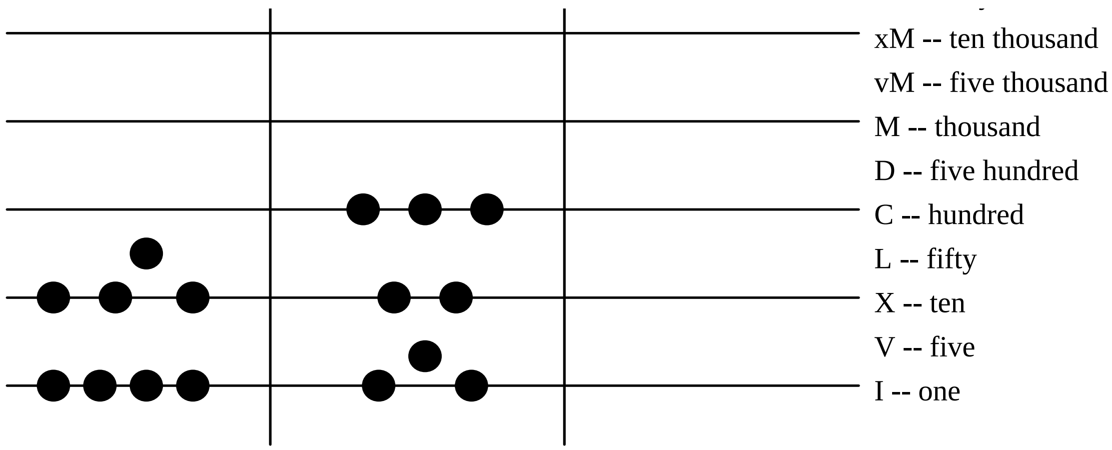

Figure 20

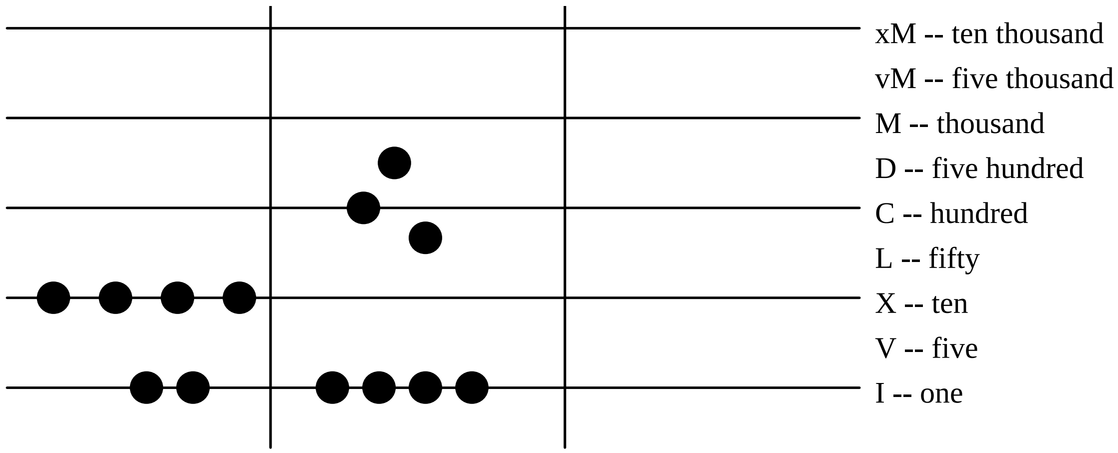

Figure 21

Since the left number is even, repeat the halving and doubling.

Since the result has an odd number on the left, copy the middle number on the right.

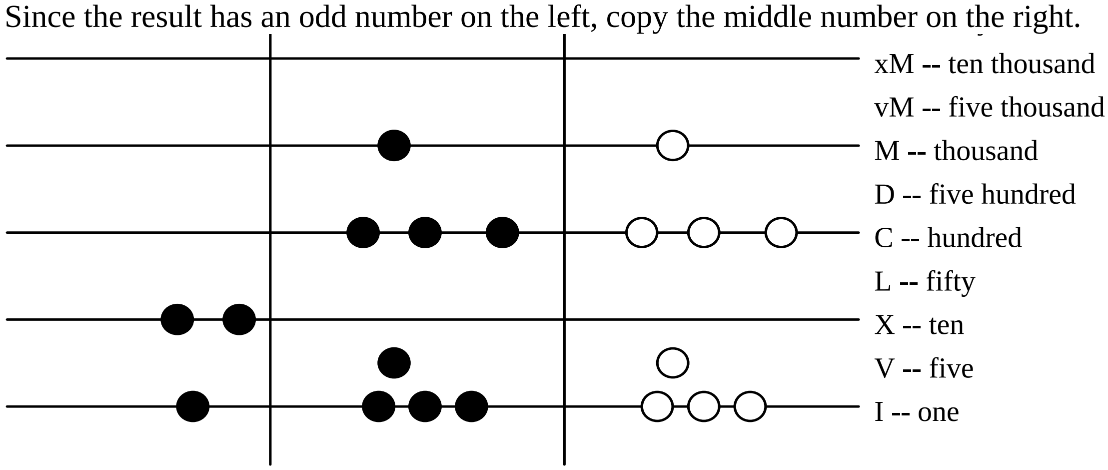

Figure 22

Remove the odd counter on the left.

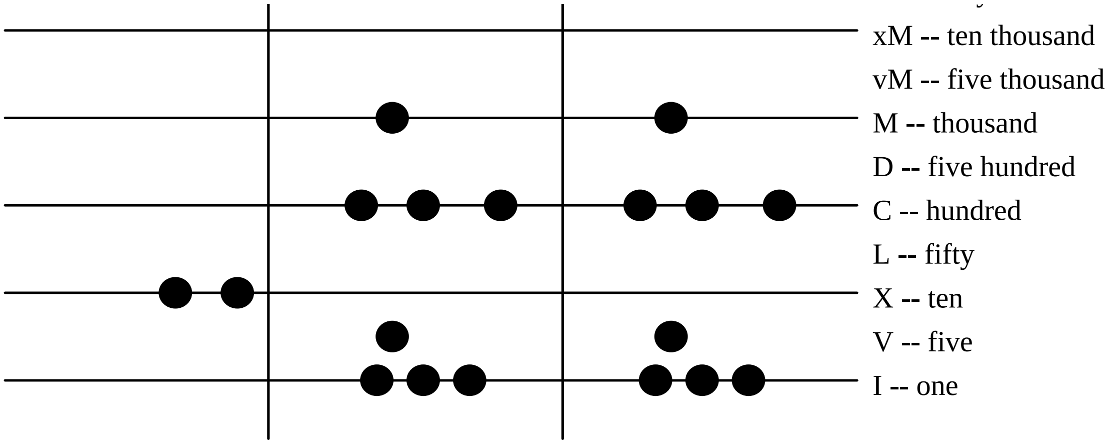

Figure 23

Halve the number on the left and double the number in the middle.

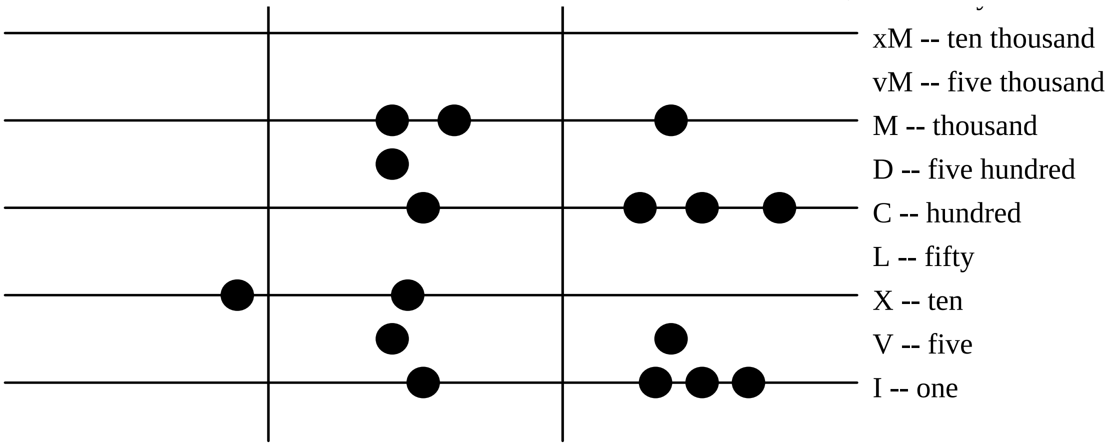

Figure 24

Since the left number is even, halve the number on the left and double the number in the middle. Since the result is odd, copy the middle number on the right.

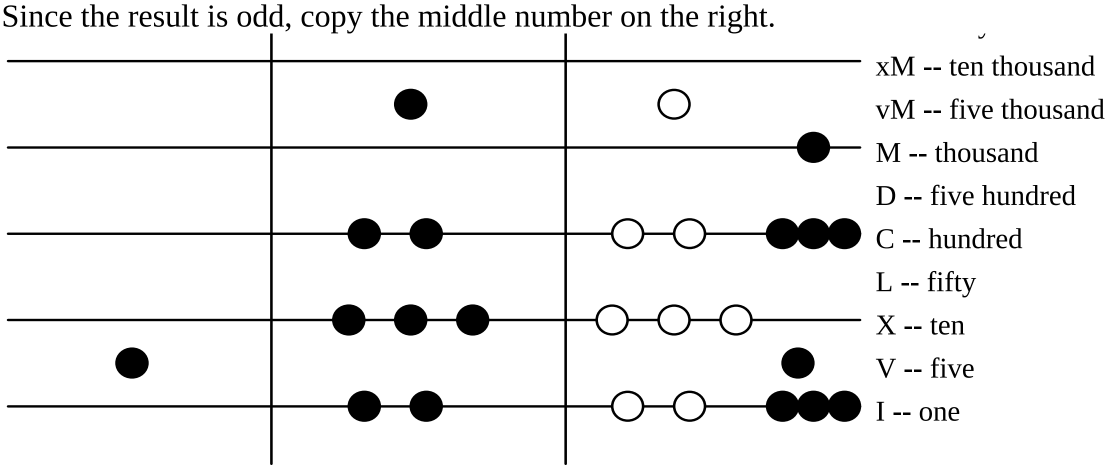

Figure 25

Remove the odd counter on the left and regroup on the right.

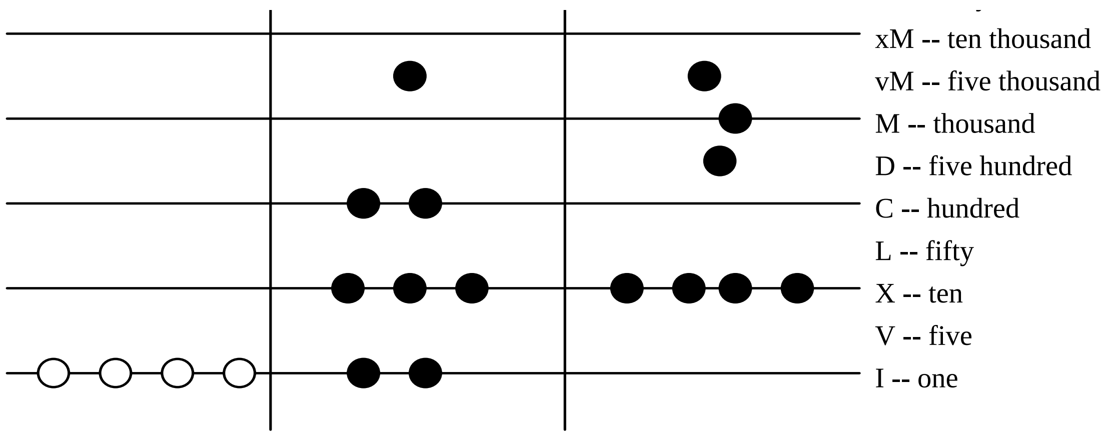

Figure 26

Halve the left and double the middle.

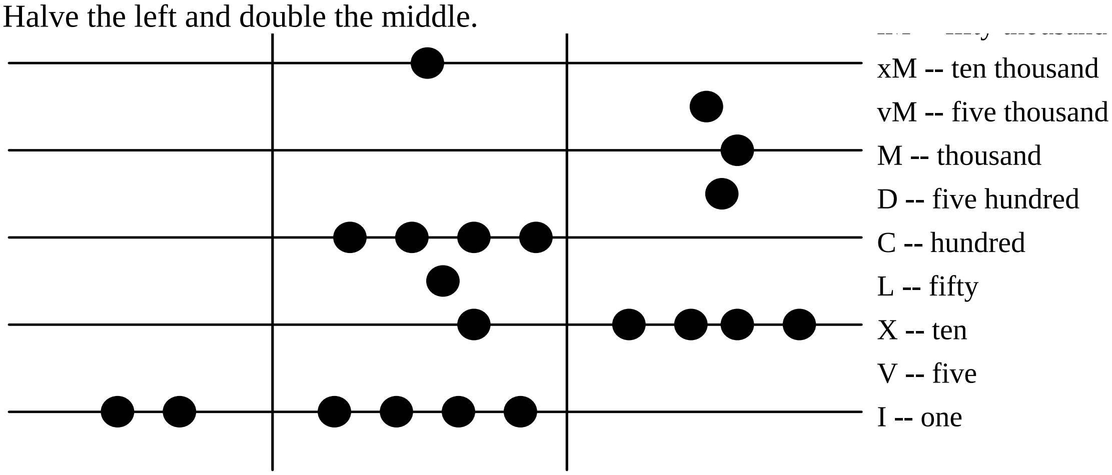

Figure 27

Since the number on the left is even, halve and double again. Copy the middle number on the right.

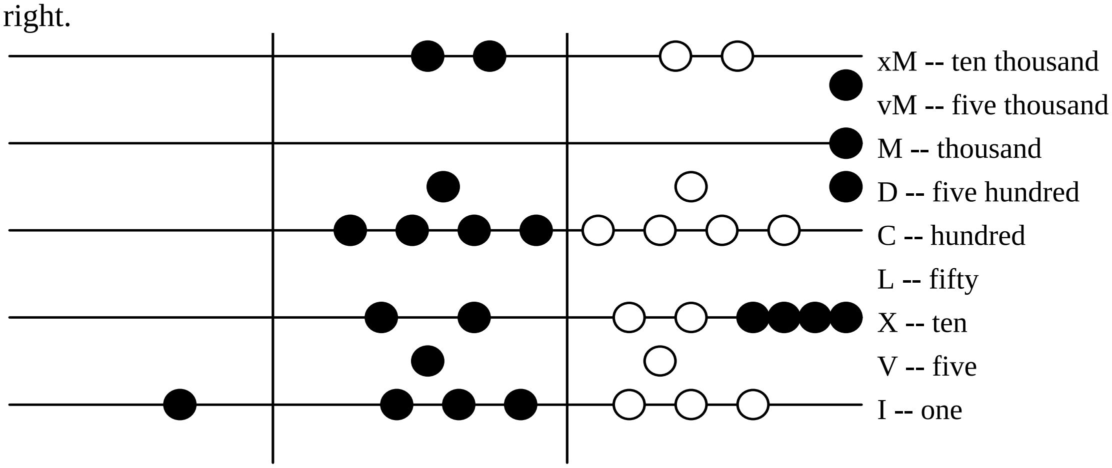

Figure 28

Remove the (single) odd counter on the left and regroup on the right.

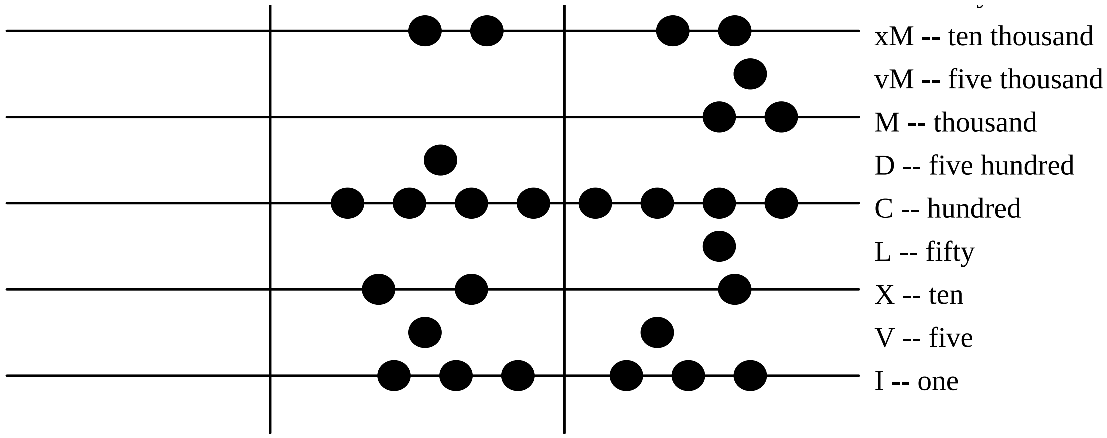

Figure 29

The result is xMxMvMMMCCCCLXVIII or 27,468.

With practice the halving and doubling can be done with a minimum of actual thought, merely as a process of moving counters around.

The astute among you will have noticed that in essence, the halving-doubling process essentially multiplies one multiplicand by the binary representation of the other.

Division

Division is the hardest of the four operations, as it is for pen-and-ink calculations. One method is very similar to the pen-and-ink method, merely using the counting board to keep the partial results.

Another method is to perform division as repeated subtraction. This method was still used in the middle of the 20th century by mechanical adding machines. We will calculate the following quotient: How many times does 29 go into 6,237? In Roman numerals this is written: How many times does XXVIIII go into vMMCCXXXVII?
Place the divisor on the board on the left, and the dividend in the middle.

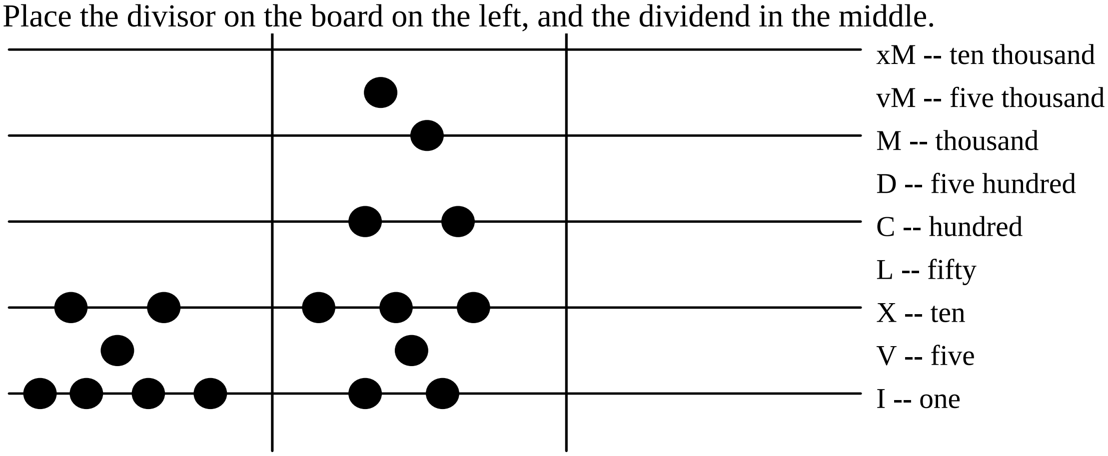

Figure 30

Take away one multiple of the divisor from the dividend, taking the lowest counters from the C line. Put one counter on the C line. This means that we have just taken away twenty-nine hundred.

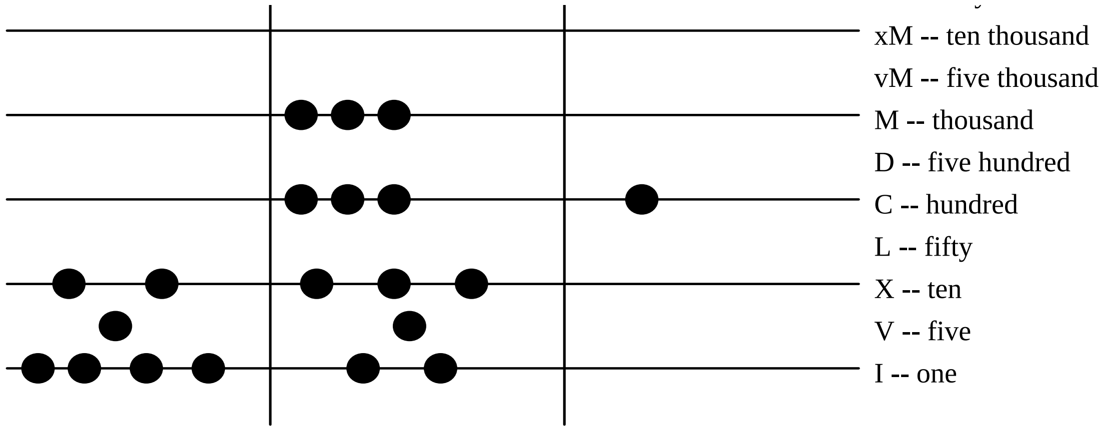

Figure 31

Repeat.

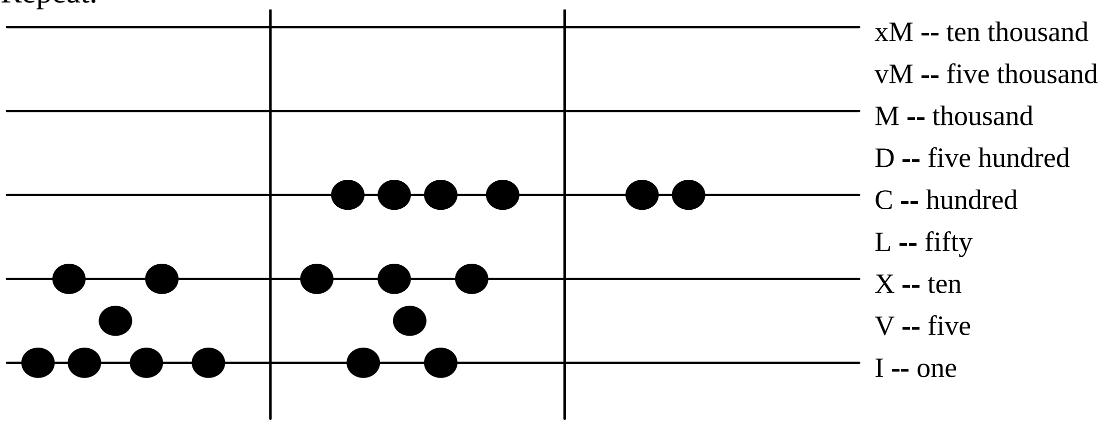

Figure 32

Since there are no counters left on the M line, we move down one line and repeat the process there.

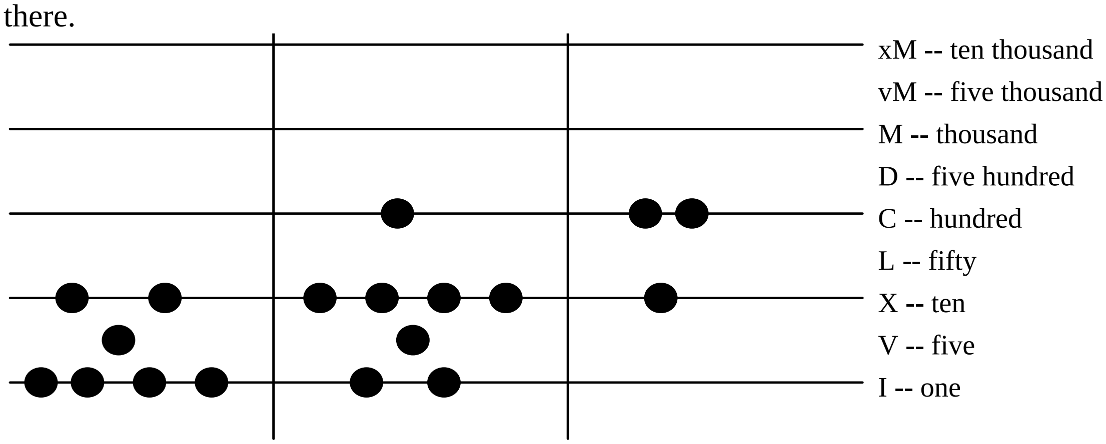

Figure 33

Since there is only one counter left on the C line, we move down one line and repeat the process there.

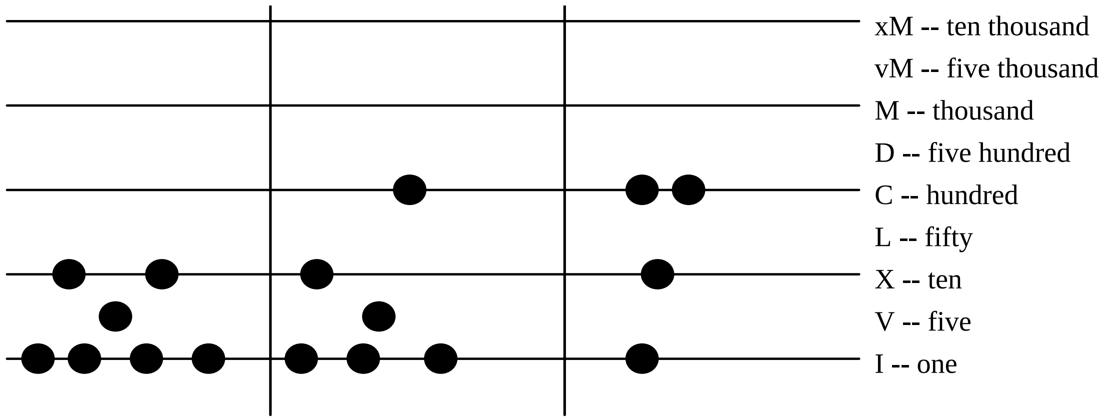

Figure 34

Repeat.

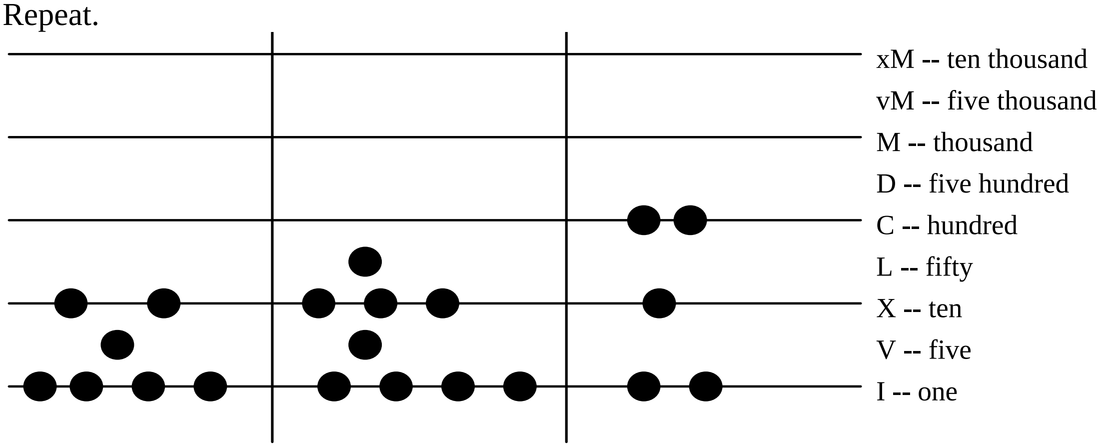

Figure 35

Repeat.

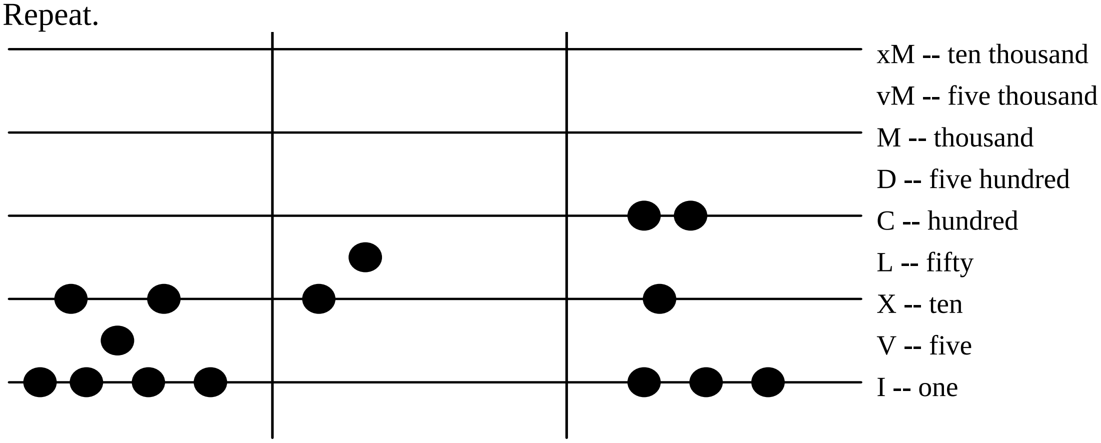

Figure 36

Repeat.

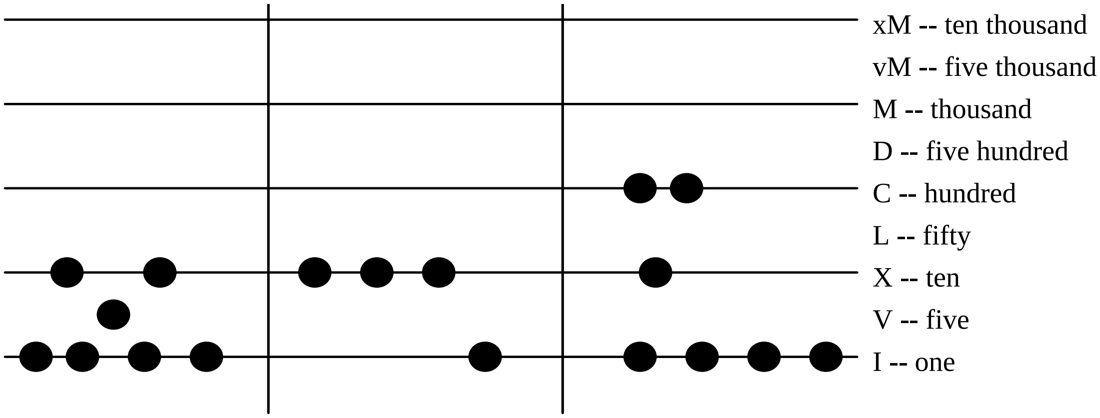

Figure 37

Repeat.

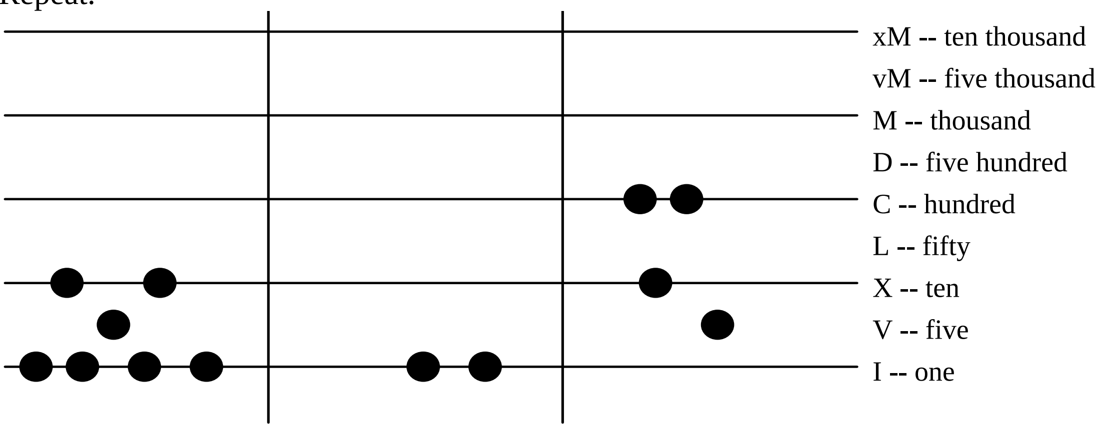

Figure 38

Since there are no counters left on or above the X line, we are done. The result is: XXVIIII goes into vMMCCXXXVII, CCXV times with a remainder of II. In Arabic figures this is 29 goes into 6,237, 215 times with a remainder of 2.

Other Calculations

The counting board/abacus can be used to perform a number of other calculations. These include square roots and cube roots. Such methods are beyond the scope of this presentation.

Bibliography

1. Mims, Forrest M. *Number machines: An introduction to the abacus, slide rule, and pocket calculator*. New York: David McKay Company, Inc. (1977).

2. Moon, Parry *The abacus: Its history; its design, its possibilities in the modern world*. New York: Gordon and Breach Science Publishers. (1971).

3. Pullan, J. M. *The history of the abacus*. New York: Frederick A. Praeger, Publishers. (1968).