AMERICAN MATHEMATICAL ASSOCIATION OF TWO YEAR COLLEGES

36 ANNUALCONFERENCE

November 11-14, 2010
Boston, MA

A Mathematics Intensive Presentation

Partial Fraction Decomposition
A Strategic Approach

Dr. Siham Alfred

Mathematics Department
Raritan Valley Community College

PARTIAL FRACTIONS DECOMPOSITION
A STRATEGIC APPROACH

The most common method in Calculus II textbooks for finding partial fraction decomposition for a rational function is to set up and solve a system of linear equations for all four cases: (1) distinct linear factors,(2) repeated linear factors, (3) distinct irreducible quadratic factors and(4) repeated irreducible quadratic factors. An example of repeated linear factors is shown below. The goal of this presentation is to explore other more strategic methods

Method 1: By Solving a System of Linear Equation:

Example 1: A Rational Function with repeated linear factors

Consider $\dfrac{x^2+4x-3}{(x-2)^3}=\dfrac{A}{x-2}+\dfrac{B}{(x-2)^2}+\dfrac{C}{(x-2)^3}$ (1)

$$x^2+4x-3=A(x-2)^2+B(x-2)+C \quad (2)$$

Collecting like terms and equation coefficients, to obtain

$$x^2+4x-3=Ax^2+(B-4A)x+4A-2B+C \quad (3)$$

which gives the following system

A = 1
-4A +B = 4
4A-2B+C = -3

Solving the system yields the values A = 1, B = 8 and C = 9

Method 2: By Assigning Values for x and Solving for the A, B and C Constants
(Larson Hostetler Edwards, seventh edition p. 515)

When substituting x = 2 in equation (2) of Example1, one obtains

4+8 -3 = C = 9

When substituting x = 1 in equation (2) one obtains

1+4 -3 =A –B + C substitution for C = 9 and collecting like terms we get
A – B = -7

When x = 0 in equation (2) one obtains

-3 = 4A -2B +C which gives -12 = 4A -2B; and -6 = 2A –B

From the system A – B = -7 and
2A –B = -6 one obtains that A = 1 and B = 8 as above.

Method 3: By Taking the Derivative of Both Sides
(Thomas' Calculus, 11th edition p. 545)

Start by taking the first derivative of equation (2) from the previous example

$$x^2+4x-3=A(x-2)^2+B(x-2)+C \qquad (2)$$

$2x+4=2A(x-2)+B$. When x = 2 then B = 8

Taking the second derivative gives $2=2A$ and A = 1. Substituting for A and B in the constant term of equation (3) one gets, 4A – 2B + C = -3 and C = 9 as above

Method 4: By the Heaviside "Cover Up" Method
(Thomas' Calculus, 11th edition p. 560)

When the degree of N(x) the numerator polynomial is less than the degree of D(x) the denominator polynomial and when the linear factors are distinct.

Example 2 Let $$\frac{x^2+1}{(x-1)(x-2)(x-3)}=\frac{A}{(x-1)}+\frac{B}{(x-2)}+\frac{C}{(x-3)} \qquad (4)^*$$

Multiply both sides of the equation (4) by (x - 1) and then let x = 1,

$$\frac{x^2+1}{(x-2)(x-3)}=A+\frac{B(x-1)}{(x-2)}+\frac{C(x-1)}{(x-3)} \text{ and } \frac{1^2+1}{(1-2)(1-3)}=A+0+0 \text{ and A = 1}$$

Similarly from equation (4) one can obtain the values for B by multiplying by (x-2) and letting x = 2

$$\frac{x^2+1}{(x-1)(x-3)}=\frac{A(x-2)}{(x-1)}+B+\frac{C(x-2)}{(x-3)} \text{ and } \frac{2^2+1}{(2-1)(2-3)}=0+B+0 \text{ and B = -5}$$

By multiplying by (x-3) and letting x = 3 one can solve for C in equation (4)

$$\frac{x^2+1}{(x-1)(x-2)} = \frac{A(x-3)}{(x-1)} + \frac{B(x-3)}{(x-2)} + C \text{ and } \frac{3^2+1}{(3-1)(3-2)} = 0+0+C \text{ and C = 5}$$

*Some may think that this Heaviside method is illegitimate because the equation should not admit x =1, x=2 or x =3 respectively for substitution. But if one starts with
$x^2+1 = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2)$ and then divide both sides by $(x-1)(x-2)(x-3)$ to get equation (4) then all is well.

<u>Method 5: By Partial Fraction Substitution (David Rose)</u>
(David Rose, The College Mathematics Journal, vol. 38, No. 2 March 2007 p. 145-147)

David Rose gives a quick method to find the partial fraction decomposition of a rational function in the special two cases when the denominator is a single repeated linear factor such as $D(x) = (ax+b)^k$ or an irreducible repeated quadratic factor such as $D(x) = (ax^2+bx+c)^k$ with $4ac > b^2$. In most of the examples below a = 1 that is the linear and quadratic polynomials are monic.

A. The Linear Case: When $D(x) = (x+b)^k$ where D(x) is the denominator

polynomial of the rational function $R(x) = \frac{N(x)}{D(x)}$. One makes the substitution

t = x + b, then x = t – b and $R(x) = \frac{N(x)}{(x+b)^k} = \frac{G(t)}{t^k}$.

Previous Example 1 and Examples 3 and 4 below are examples of this first Linear Case

Consider Example 1 again: $\frac{x^2+4x-3}{(x-2)^3}$ (5)

Rose makes the following substitution. Let $t = x - 2$, then $x = t + 2$. Substitute for x in terms of t in the problem and simplify to get

$$\frac{(t+2)^2+4(t+2)-3}{t^3} = \frac{t^2+8t+9}{t^3} = \frac{1}{t} + \frac{8}{t^2} + \frac{9}{t^3}.$$ Rewrite in terms of x

$$\frac{x^2+4x-3}{(x-2)^3} = \frac{1}{(x-2)} + \frac{8}{(x-2)^2} + \frac{9}{(x-2)^3}$$ Now one is ready to integrate

Example 3: $\dfrac{x^4+2x^3-x^2+5}{(2x-1)^5}$. Factor the 2 from the denominator and let

t = $x-\dfrac{1}{2}$ and $x=t+\dfrac{1}{2}$ to get: $\dfrac{x^4+2x^3-x^2+5}{2^5(x-\frac{1}{2})^5}=\dfrac{(t+\frac{1}{2})^4+2(t+\frac{1}{2})^3-(t+\frac{1}{2})^2+5}{32t^5}$

which, when simplified yields: $\dfrac{1}{32}\left(\dfrac{1}{t}+\dfrac{4}{t^2}+\dfrac{7/2}{t^3}+\dfrac{1}{t^4}+\dfrac{81/16}{t^5}\right)$. Writing the expression

back in x one gets: $\dfrac{1/16}{(2x-1)}+\dfrac{1/2}{(2x-1)^2}+\dfrac{7/8}{(2x-1)^3}+\dfrac{1/2}{(2x-1)^4}+\dfrac{81/16}{(2x-1)^5}$. One is ready to integrate.

Example 4: is an example where the rational function is improper.

$R(x)=\dfrac{2x^5-x^3+x-4}{(x+2)^3}$. Again here let t = x +2 so x = t – 2. Divide and substitute to get:

$$\frac{2(t-2)^5-(t-2)^3+(t-2)-4}{t^3}=2t^2-20t+79-\frac{154}{t}+\frac{149}{t^2}-\frac{62}{t^3}$$

Writing everything back in x to get $2x^2-12x+47-\dfrac{154}{(x+2)}+\dfrac{149}{(x+2)^2}-\dfrac{62}{(x+2)^3}$

The second case Rose discusses is the irreducible quadratic case

B. The Irreducible Quadratic Case: When $D(x)=(x^2+bx+c)^k$.

Step 1: Complete the square to get: $D(x)=(x^2+bx+c)^k=[(x+p)^2+q]^k$
Step 2: Make the substitution t = x + p to get $(t^2+q)^k$
Step 3: Make the substitution $s=t^2+q$

Rose gives the following example

Example 5: $R(x)=\dfrac{4x^5-17x^4+45x^3-58x^2+48x-8}{(x^2-2x+3)^3}$

Step 1: Complete the square: $R(x)=\dfrac{4x^5-17x^4+45x^3-58x^2+48x-8}{[(x-1)^2+2]^3}$

Step 2: Make the substitution t = x-1 to get $R(t) = \dfrac{4t^5 + 3t^4 + 17t^3 + 15t^2 + 19t + 14}{[t^2 + 2]^3}$.

$$Let\ R(t) = \frac{N(t)}{[t^2 + 2]^3}$$

To simplify calculations, divide three successive times by $s = t^2 + 2$

First division by $s = t^2 + 2$: $\dfrac{N(t)}{[t^2 + 2]} = 4t^3 + 3t^2 + 9t + 9 + \dfrac{t-4}{[t^2 + 2]}$

Second Division by $s = t^2 + 2$:

$$\frac{N(t)}{[t^2 + 2]^2} = \frac{4t^3 + 3t^2 + 9t + 9}{[t^2 + 2]} + \frac{t-4}{[t^2 + 2]^2} = 4t + 3 + \frac{t+3}{[t^2 + 2]} + \frac{t-4}{[t^2 + 2]^2}$$

Third Division by $s = t^2 + 2$

$$\frac{N(t)}{[t^2 + 2]^3} = \frac{4t+3}{[t^2 + 2]} + \frac{t+3}{[t^2 + 2]^2} + \frac{t-4}{[t^2 + 2]^3}$$

Step 3: make the substitution $s = t^2 + 2$ to get $\dfrac{4t+3}{s} + \dfrac{t+3}{s^2} + \dfrac{t-4}{s^3}$

Rewrite in terms of x where t = x − 1 to get

$$R(x) = \frac{4x-1}{(x^2 - 2x + 3)} + \frac{x+2}{(x^2 - 2x + 3)^2} + \frac{x-5}{(x^2 - 2x + 3)^3}$$

That example was long!

Here are some additional problems to consider:

A. Use the Method of taking the derivative to find values of the coefficients then integrate

$$\boldsymbol{\frac{x+6}{(x-2)^3}}$$

B. Use the Heaviside Method to find the coefficients then integrate the resultant expression

$$\frac{x^2 - 3}{(2x-1)(x+4)(x-5)}$$

C. Use David Rose's Substitution Method to find the partial fraction decomposition of

1. $\dfrac{x^3 - 3x^2 + 4x - 7}{(x+1)^4}$

2. $\dfrac{3x^2 - 4x + 5}{(3x-1)^3}$

3. $\dfrac{x^5 - 2x^3 + 3x^2 - 11}{(x-1)^3}$

4. $\dfrac{3x^5 - 16x^4 + 42x^3 - 27x^2 + 36x - 4}{(x^2 - 2x + 2)^3}$

How I use this material in Calculus II

1. I assign it as a project

2. I assign it as extra credit for enrichment

3. I require it from students who are enrolled in the Calculus Honors Option Program

In all the above, I require a Summary Statement**.**

Summary Statement

a. What did you learn from this assignment? .
b. Which question(s) did you find challenging?
c. How did you overcome the challenge?
d. Which is your favorite method and why?