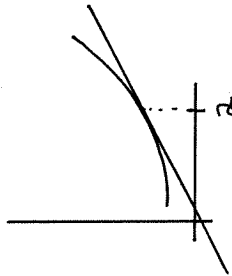


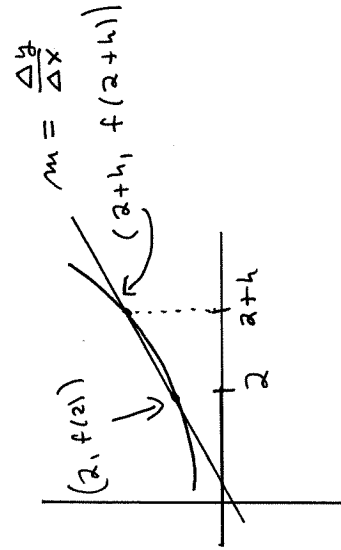
Definition of the Derivative.

Find the slope of the tangent to the graph of $y = f(x)$ at the point where $x = 2$.



Questions?

1. What is a tangent?
2. How can we compute slope?
3. Can we find approximate solution?



$$m_{\text{secant}} = \frac{\Delta y}{\Delta x} = \frac{f(2+h) - f(2)}{(2+h) - 2} = \frac{f(2+h) - f(2)}{h}$$

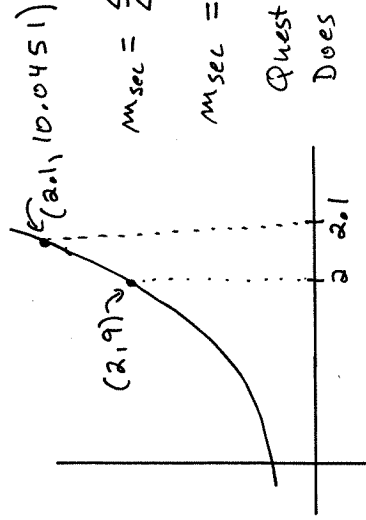
$$m_{\text{tangent}} = \lim_{h \rightarrow 0} m_{\text{secant}} = \lim_{h \rightarrow 0} \frac{f(2+h) - f(2)}{h}$$

Notation:

$f'(2)$ means the slope of the tangent to the graph of $y = f(x)$ at the point where $x = 2$.

$$\text{so } f'(2) = m_{\text{tangent}} = \lim_{h \rightarrow 0} \frac{f(2+h) - f(2)}{h}$$

Example: Let $f(x) = 3^x$. Approximate $f'(2)$



$$m_{\text{sec}} = \frac{\Delta y}{\Delta x} = \frac{10.0451 - 9}{0.1}$$

$$m_{\text{sec}} = 10.45$$

Question:

Does $f'(2) = 10.45$?

Question: What does $f'(x)$ mean?

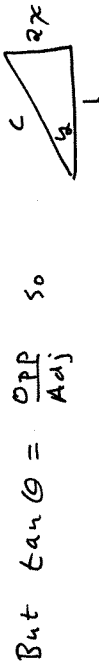
Question: What type of graph would not have the slope of a tangent at $x = 2$?

Derivatives of Inverse Functions:

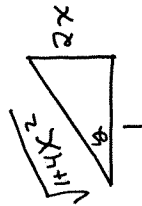
Let $f(x) = \tan^{-1} 2x$. Find $f'(x)$.

$y = \tan^{-1} 2x$. Goal: Find $\frac{dy}{dx}$

If $y = \tan^{-1} 2x$, then $\tan y = 2x$



$$1^2 + (2x)^2 = c^2 \Rightarrow 1 + 4x^2 = c^2 \Rightarrow \sqrt{1 + 4x^2} = c$$



$$\tan y = 2x$$

$$\frac{d}{dx} \tan y = \frac{d}{dx} 2x \rightarrow \text{But } \cos y = \frac{\text{ADJ}}{\text{HYP}}$$

$$\sec y \frac{dy}{dx} = 2 \quad \left. \begin{array}{l} \cos y = \frac{1}{\sqrt{1+4x^2}} \\ \sec y = \sqrt{1+4x^2} \\ \sec^2 y = 1+4x^2 \end{array} \right\}$$

$$\frac{dy}{dx} = \frac{2}{1+4x^2}$$

$$\frac{dy}{dx} = \frac{2}{1+4x^2}$$

So when $f(x) = \tan^{-1} 2x$

$$f'(x) = \frac{2}{1+4x^2}$$

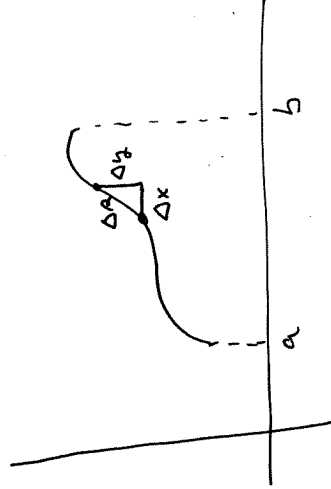
Let $y = f^{-1}(x)$. Find y' .

$$f(y) = x \Rightarrow \frac{d}{dx} f(y) = x \Rightarrow f'(y) \frac{dy}{dx} = 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{f'(y)} \Rightarrow \frac{dy}{dx} = \frac{1}{f'(f^{-1}(x))}$$

Arc-Length

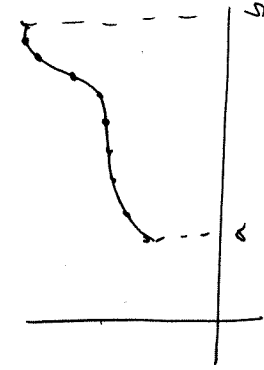
Find the arc length of $y = f(x)$ on $[a, b]$



From Pythagoras we know $(\Delta x)^2 + (\Delta y)^2 = (\Delta A)^2$

$$\text{So } \Delta A = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

Now break up the graph into many small parts



So Total arc length =

$$\sum \Delta A$$

$$A = \sum \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

$$A = \sum \sqrt{\frac{(\Delta x)^2 + (\Delta y)^2}{(\Delta x)^2}} \Delta x = \sum \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2} \Delta x$$

Shrink Δx and Δy to 0.

$$A = \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

Taylor and Maclaurin Series

Find a Maclaurin series for $f(x) = e^{2x}$

Goal: Find a polynomial $p(x)$ that approximates $f(x) = e^{2x}$.

$$p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \dots$$

$$f(x) = e^{2x}$$

We want $p(0) = f(0)$, $p'(0) = f'(0)$, $p''(0) = f''(0)$ etc.

$$p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \dots$$

$$p'(x) = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 + \dots$$

$$p''(x) = 2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + \dots$$

$$p'''(x) = 6a_3 + 24a_4x + 60a_5x^2 + \dots$$

$$p^{(4)}(x) = 24a_4 + 120a_5x + \dots$$

$$p^{(5)}(x) = 120a_5 + \dots$$

$$p(0) = a_0 \quad p'(0) = a_1 \quad p''(0) = 2a_2 \quad p'''(0) = 6a_3$$

$$p^{(4)}(0) = 24a_4 \quad p^{(5)}(0) = 120a_5 \quad p^{(6)}(0) = ?$$

$$f(x) = e^{2x} \quad f(0) = 1$$

$$f'(x) = 2e^{2x} \quad f'(0) = 2$$

$$f''(x) = 4e^{2x} \quad f''(0) = 4$$

$$f'''(x) = 8e^{2x} \quad f'''(0) = 8$$

$$f^{(4)}(x) = 16e^{2x} \quad f^{(4)}(0) = 16$$

$$f^{(5)}(x) = 32e^{2x} \quad f^{(5)}(0) = 32$$

$$p(0) = f(0) \quad p^{(4)}(0) = f^{(4)}(0)$$

$$a_0 = 1 \quad 24a_4 = 16$$

$$p'(0) = f'(0) \quad a_4 = \frac{16}{24} = \frac{2}{3}$$

$$a_1 = 2 \quad p^{(5)}(0) = f^{(5)}(0)$$

$$p''(0) = f''(0) \quad 120a_5 = 32$$

$$2a_2 = 4 \quad a_5 = \frac{32}{120} = \frac{4}{15}$$

$$a_2 = 2$$

$$p^{(6)}(0) = f^{(6)}(0)$$

$$6a_3 = 8 \quad 6!a_6 = 64$$

$$a_3 = \frac{8}{6} = \frac{4}{3} \quad a_6 = \frac{64}{6!} = \frac{4}{45}$$

$$\text{So } f(x) = e^{2x} \approx 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4 +$$

$$\frac{4}{15}x^5 + \frac{4}{45}x^6 + \dots$$

Is there a pattern?

$$p^{(N)}(0) = N! a_N$$

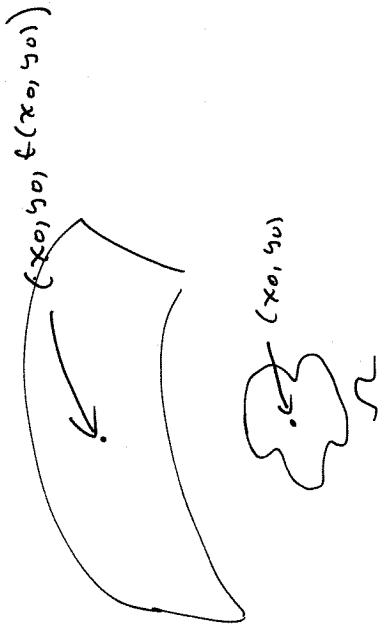
We need $p^{(N)}(0) = f^{(N)}(0)$ so

$$N! a_N = f^{(N)}(0)$$

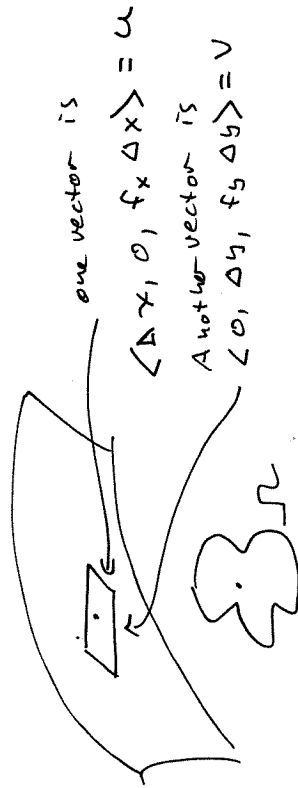
$$a_N = \frac{f^{(N)}(0)}{N!}$$

Surface Area

Find the area of the surface $z = f(x, y)$ above the region R .



"Glue" a small piece of "paper-mache" that represents the tangent plane to (x_0, y_0) at the point $(x_0, y_0, f(x_0, y_0))$.



What is the area of that piece of paper-mache?

Answer: $\|u \times v\|$

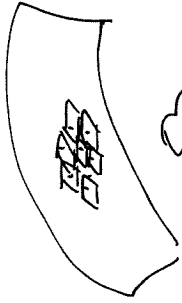
$$u \times v = \det \begin{bmatrix} i & j & k \\ \Delta x & 0 & f_x \Delta x \\ 0 & \Delta y & f_y \Delta y \end{bmatrix}$$

$$u \times v = \langle -f_x \Delta x \Delta y, -f_y \Delta x \Delta y, \Delta x \Delta y \rangle$$

$$= |\Delta x \Delta y| \langle -f_x, -f_y, 1 \rangle$$

$$\|u \times v\| = \sqrt{f_x^2 + f_y^2 + 1} \Delta x \Delta y$$

The total surface area is the sum of all the pieces of paper mache over the region R .



$$\cdot SA \approx \sum \sqrt{f_x^2 + f_y^2 + 1} \Delta x \Delta y$$

Shrink Δx and Δy to zero, and get

$$SA = \iint_R \sqrt{f_x^2 + f_y^2 + 1} \, dx \, dy$$